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Sharp bounds on the moments of linear combinations of order statistics and kth records

PhD dissertation

Optymalne oszacowania momentów kombinacji liniowych statystyk pozycyjnych i k-tych rekordów

rozprawa doktorska

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Author's declaration:
aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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Supervisor's declaration:
the dissertation is ready to be reviewed.

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to my mother

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Preface

This dissertation is devoted to determining sharp bounds on the expectations and variances of linear combinations of order statistics and kth records based on independent and identically distributed random variables. Order statistics arise in a natural way by ordering random variables in the ascending order. Classic first upper record values are the observations that are greater than all the preceding ones. Their extensions, so called kth upper records, are the new values appearing at the kth upper position in the sequence of ordered observations. Order statistics and their linear combinations, called L-statistics, play a vital role in statistical inference. Moreover, they are extensively used in survival analysis, reliability theory, and treatment of censored data. Record values are applied for prediction of catastrophes, and extreme events in nature and sports.

Many evaluations of expectations of linear combinations of order and record statistics were presented in the literature. The novelty of our results consists in measuring the respective bounds in the scale units being the Gini mean difference of the population. The Gini mean difference of a probability distribution is the expectation of the absolute value of the difference of two independent copies of random variables with the parent distribution. The Gini mean difference becomes a popular and useful measure of dispersion. One of its virtues is that it can be defined under assumption of finiteness of the first population moment only (note that the standard deviation which is the most popular scale unit requires existence of the second moment). We prove that our bounds on the expectations of linear combinations of order and record statistics are sharp, and describe conditions of their attainability.

Much less is by now about bounds on variances of ordered random variables. Our bounds on variances of linear combinations of order statistics and kth record values are expressed in variance units of the original i.i.d. observations. Until now only bounds on variances of single order statistics and single kth records were presented in literature. We describe bounds on variances of arbitrary linear combinations of order and record statistics, and present conditions of their attainability. We also specify general results for single order and record statistics and their increments.

The main idea of our reasoning consists in integral representation of the expectations, variances and covariances of order and record statistics so that the integrand is the composition of some (usually complicated) function with the baseline distribution function. The thesis is organized as follows.

Chapter 1 contains some essential information which is used in the next chapters. We present some distributional properties of order statistics and kth record values. Moreover, variation diminishing property (VDP, for short) of some families of functions is also introduced in this chapter. The property asserts that a linear combination of a sequence of functions has no more sign changes than the respective sequence of combination coefficients. This is a useful tool in our studies.

In Chapter 2, it is provided a method of calculating sharp lower and upper bounds on the expectations of arbitrary, properly centered L-statistics expressed in the Gini mean difference units of the original i.i.d. observations. Precise values of bounds are derived for the single order statistics, their differences, and some most popular examples of L-statistics such as: the trimmed means, Winsorized means, and mean absolute deviation from the median. It also presents the families of discrete distributions which attain the bounds, possibly in the limit. This chapter is based on the paper by Kozyra and Rychlik (2017a).

In Chapter 3 we first describe the idea of obtaining lower and upper bounds on the variances of arbitrary linear combinations of order statistics and sufficient conditions of their attainability. Then we provide tight bounds for some special cases. We remind the results of Papadatos (1995) who presented sharp lower and upper bounds on the variances of single order statistics expressed in the population variance units. Then we determine analogous results for spacings, i.e. differences of consecutive order statistics. Finally, we determine the upper bounds on the variances of linear combinations spacings based on three observations. This example shows that establishing optimal bounds for general L-statistics is actually a challenging task. The most of the results of this chapter were presented in Kozyra and Rychlik (2017b).

Chapter 4 is entirely based on the paper by Kozyra and Rychlik (2017c). Here we describe a method of calculating sharp lower and upper bounds on the expectations of linear combinations of kth records expressed in the Gini mean difference units of parent distribution. In particular, we provide sharp lower and upper bounds on the expectations of kth records and their differences. We also present the families of distributions which attain the bounds in the limit.

Chapter 5 is devoted to the study of bounds on the variances of linear combinations of kth record values. Some upper evaluations are presented, together with conditions of their sharpness. We also point out assumptions under which the lower variance bounds trivially become zero. Then some special cases are treated. We cite results of Klimczak and Rychlik (2004) where sharp bounds on variances of single kth record values were presented. They were more precisely specified by Jasiński (2016). Then we provide similar sharp bounds for the kth record spacings which are defined as the differences between adjacent kth record values. The results of the chapter were earlier presented in Kozyra and Rychlik (2017d).

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Notation

$\mathbb{N}$	—	set of natural numbers ($0 \notin \mathbb{N}$)
$\mathbb{R}$	—	set of real numbers
i.i.d.	—	independent identically distributed
VDP	—	variation diminishing property
$F(x)$	—	distribution function
μ	=	$\mathbb{E}X$ — expectation of random variable X
$\mathbb{V}arX$	—	variance of random variable X
$\mathbb{C}ov(X,Y)$	—	covariance of random variables X and Y
Δ	=	$\mathbb{E}\|X_1 - X_2\|$ — Gini mean difference (X_1, X_2 are i.i.d.)
$X_{k:n}$	—	kth order statistic based on n random variables $X_1, \ldots, X_n$
$S_{i:n}$	=	$X_{i+1:n} - X_{i:n}$ — ith spacing of order statistics based on $X_1, \ldots, X_n$
$F^X_{k:n}(x)$	—	marginal distribution function of $X_{k:n}$ based on i.i.d. random variables $X_1, \ldots, X_n$ with general distribution function F, see (1.1.1), p. 2
$F^X_{i,j:n}(x,y)$	—	joint distribution function of $X_{i:n}$ and $X_{j:n}$ based on i.i.d. random variables $X_1, \ldots, X_n$ with general distribution function F, see (1.1.2), p. 3
$F^U_{k:n}(u)$	—	marginal distribution function of kth order statistic based on n i.i.d. standard uniform random variables
$F^U_{i,j:n}(u,v)$	—	joint distribution function of ith and jth order statistics based on n i.i.d. standard uniform random variables
$B_{k,m}(u)$	=	$\binom{m}{k}u^k(1-u)^{m-k}$, $0 < u < 1$, $k = 0, \ldots, m$, — kth Bernstein polynomial of degree m
$\binom{n}{a,b}$	=	$\frac{n!}{a!b!(n-a-b)!}$ — trinomial coefficient
$R_{n,k}$	—	nth value of kth (upper) record
$F^X_{n,k}(x)$	—	marginal distribution function of nth value of kth record based on i.i.d. sequence $X_1, X_2, \ldots$ with general continuous distribution function F, see (1.2.5), p. 6

$F^X_{m,n,k}(x,y)$	—	joint distribution function of mth and nth values of kth records based on i.i.d. sequence $X_1, X_2, \ldots$ with general continuous distribution function F, see (1.2.6), p. 6
$F^U_{n,k}(u)$	—	marginal distribution function of nth value of kth record based on i.i.d. standard uniform sequence
$F^U_{m,n,k}(u,v)$	—	joint distribution function of mth and nth value of kth record based on i.i.d. standard uniform sequence
$\Xi_{\mathbf{c}}(u)$	—	see (2.1.2), p. 14
$\Xi_{r:n}(u)$	—	see (2.2.1)–(2.2.5), p. 18–19
$\Xi_{r,s:n}(u)$	=	$\Xi_{s:n}(u) - \Xi_{r:n}(u)$ — see (2.3.1), p. 21
$\Phi_{\mathbf{c}}(u,v)$	—	see (3.1.1), p. 38
$\Psi_{\mathbf{c}}(u)$	=	$\Phi_{\mathbf{c}}(u,u)$ — see (3.1.2), p. 38
$\Phi_{i:n}(u,v)$	—	see (3.3.1), p. 42
$\Psi_{i:n}(u)$	=	$\Phi_{i:n}(u,u)$ — see (3.3.2), p. 42
$\xi_{n,k}(u)$	—	see (4.1.2), p. 56
$\Xi_{n,k}(u)$	=	$\frac{\xi_{n,k}(u)}{2u}$, see (4.1.3), p. 56
$\xi_{\mathbf{c},k}(u)$	—	see (4.1.4), p. 56
$\Xi_{\mathbf{c},k}(u)$	=	$\frac{\xi_{\mathbf{c},k}(u)}{2u}$, see (4.1.5), p. 56
$\Xi_{m,n;k}(u)$	=	$\Xi_{n,k}(u) - \Xi_{m,k}(u)$, see (4.3.1), p. 67
$\Phi_{\mathbf{c},k}(u,v)$	—	see (5.1.1), p. 76
$\Psi_{\mathbf{c},k}(u)$	=	$\Phi_{\mathbf{c},k}(u,u)$ — see (5.1.2), p. 76
$\Phi_{m,k}(u,v)$	—	see (5.3.1), p. 81
$\Psi_{m,k}(u)$	=	$\Phi_{m,k}(u,u)$
$\psi_{m,k}(u)$	=	$u\Psi_{m,k}(u)$

Chapter 1

Preliminaries

In this chapter we define order statistics and kth record values. In the cases when they are based on independent and identically distributed (i.i.d., for brevity) random variables, we determine their one- and two-dimensional marginal distribution functions. Then we use the respective formulae for establishing integral representations of variances and covariances of order and record statistics. Finally we describe so called variation diminishing property of selected sequences of functions. The property is frequently used in our further analysis.

1.1 Order statistics

1.1.1 Definition, L-statistics, and spacings

Consider n variables $X_1, \dots, X_n$ defined on the same probability space $(\Omega, \mathbb{F}, \mathbf{P})$. If we arrange these variables in increasing order, we obtain order statistics $X_{1:n} \leq \dots \leq X_{n:n}$. Linear combinations of order statistics $\sum_{i=1}^{n} c_i X_{i:n}$ with fixed real coefficients $c_1, \dots, c_n$ is called L-statistics. L-statistics are widely applied in statistical inference. For instance, the trimmed and Winsorized means are used for estimating location of populations, whereas mean absolute deviation from the median and sample range are popular measures of scale. Other useful examples of L-statistics are spacings defined as $S_{i:n} = X_{i+1:n} - X_{i:n}$ for $i \in \{1, \dots, n-1\}$.

1.1.2 Distribution functions

Now we consider n i.i.d. random variables $X_1, \dots, X_n$ with common distribution function F. It is obvious that for any $x \in \mathbb{R}$:

$$F_{n:n}^{X}(x) \;=\; \mathbb{P}(X_{n:n} \leq x) = \mathbb{P}(X_1 \leq x, \dots, X_n \leq x) = \prod_{i=1}^{n} \mathbb{P}(X_i \leq x) = F^n(x).$$

Similarly

$$\begin{aligned} F_{1:n}^X(x) &= \mathbb{P}(X_{1:n} \le x) = 1 - \mathbb{P}(X_{1:n} > x) \\ &= 1 - \mathbb{P}(X_1 > x, \dots, X_n > x) = 1 - \big(1 - F(x)\big)^n. \end{aligned}$$

In general case for any $k \in \{1, \dots, n\}$ and $x \in \mathbb{R}$ we have

$$\begin{aligned} F_{k:n}^X(x) &= \mathbb{P}(X_{k:n} \le x) \\ &= \mathbb{P}(\text{at least } k \text{ variables among } X_1, \dots, X_n \text{ are not greater than } x) \\ &= \sum_{m=k}^{n} \mathbb{P}(\text{exactly } m \text{ variables among } X_1, \dots, X_n \text{ are not greater than } x) \\ &= \sum_{m=k}^{n} \binom{n}{m} F^m(x)\big(1 - F(x)\big)^{n-m}. \end{aligned} \tag{1.1.1}$$

Observe that the distribution function of single order statistic can be represented with use of Bernstein polynomials of degree n

$$B_{m,n}(u) = \binom{n}{m} u^m (1-u)^{n-m}, \qquad 0 < u < 1, \quad m = 0, \dots, n,$$

as follows

$$F_{k:n}^X(x) = \sum_{m=k}^{n} B_{m,n}(F(x)).$$

Similarly we obtain the joint distribution of ith and jth order statistics from observations $X_1, \dots, X_n$ for $1 \le i < j \le n$. If $x \ge y$, then obviously

$$F_{i,j:n}^X(x, y) = \mathbb{P}\left(X_{i:n} \le x, X_{j:n} \le y\right) = \mathbb{P}\left(X_{j:n} \le y\right) = F_{j:n}^X(y).$$

If $x < y$, then

$$\begin{aligned}
F^X_{i,j:n}(x,y) &= \mathbb{P}\left(X_{i:n} \le x, X_{j:n} \le y\right) \\
&= \mathbb{P}(\text{at least } i \text{ variables among } X_1,\dots,X_n \text{ belong to } (-\infty,x] \\
&\quad \text{and at least } j \text{ variables among } X_1,\dots,X_n \text{ belong to } (-\infty,y]) \\
&= \sum_{s=j}^{n}\sum_{r=i}^{s} \mathbb{P}(\text{exactly } r \text{ variables among } X_1,\dots,X_n \text{ belong to } (-\infty,x] \\
&\quad \text{exactly } s \text{ variables among } X_1,\dots,X_n \text{ belong to } (-\infty,y]) \\
&= \sum_{s=j}^{n}\sum_{r=i}^{s} \mathbb{P}(\text{exactly } r \text{ variables among } X_1,\dots,X_n \text{ belong to } (-\infty,x] \\
&\quad \text{and exactly } s-r \text{ variables among } X_1,\dots,X_n \text{ belong to } (x,y] \\
&\quad \text{and exactly } n-s \text{ variables among } X_1,\dots,X_n \text{ belong to } (y,\infty)) \\
&= \sum_{s=j}^{n}\sum_{r=i}^{s} \binom{n}{r,\, s-r} F^r(x)\big(F(y)-F(x)\big)^{s-r}\big(1-F(y)\big)^{n-s},
\end{aligned}$$

where $\binom{n}{a,b} = \frac{n!}{a!b!(n-a-b)!}$. Summing up, we have

$$F^X_{i,j:n}(x,y) = \begin{cases} \sum_{s=j}^{n}\sum_{r=i}^{s} \binom{n}{r,s-r} F^r(x)\big(F(y)-F(x)\big)^{s-r}\big(1-F(y)\big)^{n-s}, & x<y, \\ \sum_{m=j}^{n}\binom{n}{m} F^m(y)\big(1-F(y)\big)^{n-m}, & x \ge y. \end{cases} \tag{1.1.2}$$

Clearly, we obtain analogous distribution functions $F^U_{k:n}$ $F^U_{i,j:n}$ in the standard uniform case by replacing $F(x)$ and $F(y)$ in the right-hand sides of (1.1.1) and (1.1.2) by x and y, respectively, from interval $(0,1)$. The above formulae can be found in monographs, see, e.g., David and Nagaraja (2003, pp. 9 and 12) and Nevzorov (2000, pp. 6–7).

1.2 kth records values

1.2.1 Definition, record spacings

Let $X_1, X_2, \dots$ be a sequence of real random variables. The first (upper) records, introduced by Chandler (1952), are these elements of the sequence which exceed all their predecessors. More general notions, presented in Dziubdziela and Kopociński (1976), are so called (upper) kth records which are new (greater than previous ones) kth greatest elements of samples $X_1, \dots, X_n$ when n increases from k to infinity.

Precisely, for a given $k \in \mathbb{N}$, Dziubdziela and Kopociński (1976) defined the kth record times $T_{n,k}$ and the kth record values $R_{n,k}$ as follows:

$$\begin{aligned} T_{1,k} &= 1, \\ T_{n+1,k} &= \min\{j > T_{n,k}\colon X_{j:j+k-1} > X_{T_{n,k}:T_{n,k}+k-1}\}, \\ R_{n,k} &= X_{T_{n,k}:T_{n,k}+k-1}, \qquad\qquad n \in \mathbb{N}, \end{aligned}$$

where $X_{i:n}$ stands for the ith order statistic obtained from the first n observations. There is another convention of defining record times as $L(n,k) = T_{n,k} + k - 1$ in connection with the number of random variables observed till the time the respective kth record occurs (see, e.g., Nevzorov, 2000, p. 82). The choice of convention does not affect the definition of record values.

The nth spacing of kth records is defined as the nth increment of kth records $R_{n+1,k} - R_{n,k}$, $n \in \mathbb{N}$.

1.2.2 Distribution functions

From now on, we assume that random variables $X_1, X_2, \ldots$ are i.i.d. with a common continuous distribution function F. Under the assumption, the first value of first records is just the first observation X_1. It is intuitively obvious that the distribution of $R_{n+1,1}$ under condition that $R_{n,1} = x$ is identical with the distribution of the original random variable X_1 under condition that $X_1 > x$. In other words, distribution of $R_{n+1,1} - R_{n,1}$ under condition $R_{n,1} = x$ coincides with the distribution of $X_1 - x$ under condition $X_1 > x$. This implies in the case of i.i.d. standard exponential sequence $Z_1, Z_2, \ldots$ by the lack of memory of the exponential distribution that the first record value $Z_{1,1}$ and consecutive first record spacings $Z_{2,1} - Z_{1,1}, Z_{3,1} - Z_{2,1}, \ldots$ are also i.i.d. standard exponential (cf., Nevzorov, 2000, Corollary 15.7). It further follows that $Z_{m,1}$ and $Z_{n,1} - Z_{m,1}$ for any $1 \le m < n$ are independent and have Erlang (gamma) distributions with unit scale parameter and shape parameters m and $n - m$, respectively. In particular, $Z_{m,1}$ has distribution function

$$F^Z_{m,1}(x) = 1 - e^{-x} \sum_{i=0}^{m-1} \frac{x^i}{i!}, \qquad x > 0.$$

Moreover, $Z_{m,1}$ and $Z_{n,1} = Z_{m,1} + (Z_{n,1} - Z_{m,1})$ have the joint density function

$$f^Z_{m,n,1}(x,y) = \frac{x^{m-1}(y-x)^{n-m-1}e^{-y}}{(m-1)!(n-m-1)!} \qquad 0 < x < y.$$

(cf Arnold *et al*, 1998, p. 11). This allows us to calculate the joint distribution function of $Z_{m,1}$ and $Z_{n,1}$. When $x \ge y > 0$, we obtain the marginal distribution function of the latter variable

$$F^Z_{m,n,1}(x,y) = \mathbb{P}(Z_{m,1} \le x, Z_{n,1} \le y) = \mathbb{P}(Z_{n,1} \le y) = F^Z_{n,1}(y) = 1 - e^{-y}\sum_{i=0}^{n-1}\frac{y^i}{i!}. \tag{1.2.1}$$

If $0 < x < y$, then

$$\begin{aligned} F^Z_{m,n,1}(x,y) &= \int_0^x \frac{s^{m-1}}{(m-1)!}\,ds \int_s^y \frac{(t-s)^{n-m-1}}{(n-m-1)!}e^{-t}\,dt \\ &= \int_0^x \frac{s^{m-1}}{(m-1)!}\,ds \int_0^{y-s} \frac{t^{n-m-1}}{(n-m-1)!}e^{-t}\,dt \\ &= \int_0^x \frac{s^{m-1}}{(m-1)!}\left[1 - e^{s-y}\sum_{i=0}^{n-m-1}\frac{(y-s)^i}{i!}\right] ds \\ &= 1 - e^{-x}\sum_{i=0}^{m-1}\frac{x^i}{i!} - e^{-y}\sum_{i=0}^{n-m-1}\int_0^x \frac{s^{m-1}(y-s)^i}{(m-1)!i!}\,ds \\ &= F^Z_{m,1}(x) - e^{-y}\sum_{i=0}^{n-m-1}\sum_{j=0}^{i}\frac{(-1)^j x^{m+j} y^{i-j}}{(m-1)!j!(i-j)!(m+j)}. \end{aligned} \tag{1.2.2}$$

The first value of kth record is the minimum $X_{1:k}$ of first k observations $X_1, \dots, X_n$. In the i.i.d. case, under condition that $R_{n,k} = x$, the next kth record value $R_{n+1,k}$ has the distribution as the minimum of k independent copies of original variables which exceed level x. This means that $R_{1,k}, R_{2,k}, \dots$ based on an i.i.d. sequence with distribution function F have the same joint distribution as the sequence of first records based on i.i.d. sequence $\min\{X_1, \dots, X_k\}, \min\{X_{k+1}, \dots, X_{2k}\}, \dots$ with the baseline distribution function $1-(1-F)^k$ (cf. Nevzorov, 2000, Theorem 22.6). In the case of standard exponential parent distribution function F, transformation $F \mapsto 1-(1-F)^k$ leads to the exponential distribution with scale parameter $\frac{1}{k}$ which means that $Z_{1:k}$ and $\frac{Z_1}{k}$ have identical distributions. Therefore the sequences of kth records $Z_{1,k}, Z_{2,k}, \dots$ and first records $\frac{Z_{1,1}}{k}, \frac{Z_{2,1}}{k}, \dots$ divided by k are identically distributed as well. Accordingly, the one- and two-dimensional marginal distribution functions of kth records based on standard exponential sequences are

$$\begin{aligned} F^Z_{n,k}(x) &= F^Z_{n,1}(kx), \\ F^Z_{m,n,k}(x,y) &= F^Z_{m,n,1}(kx,ky). \end{aligned}$$

It is obvious that strictly increasing transformations $h(X_1), h(X_2), \dots$ of original random variables preserve strict ordering. In consequence, $h(R_{1,k}), h(R_{2,k}) \dots$ constitute kth record values in the transformed sequence $h(X_1), h(X_2), \ldots$ In particular, function $h(x) = F^{-1}(1-\exp(-x))$, where F^{-1} is the quantile function of continuous distribution function F, is strictly increasing. This implies that $F^{-1}(1-\exp(-Z_1)), F^{-1}(1-\exp(-Z_2))\ldots$ is a sequence of i.i.d. random variables with parent distribution function F, whereas $F^{-1}(1-\exp(-Z_{1,k)}), F^{-1}(1-\exp(-Z_{2,k}))\ldots$ is the corresponding sequence of kth records (see, Nevzorov, 2000, Representation 22.1). Therefore

$$F^X_{n,k}(x) = F^Z_{n,k}\Big(-\ln\big(1-F(x)\big)\Big) = F^Z_{n,1}\Big(-k\ln\big(1-F(x)\big)\Big), \tag{1.2.3}$$

$$\begin{aligned} F^X_{m,n,k}(x,y) &= F^Z_{m,n,k}\Big(-\ln\big(1-F(x)\big), -\ln\big(1-F(y)\big)\Big) \\ &= F^Z_{m,n,1}\Big(-k\ln\big(1-F(x)\big), -k\ln\big(1-F(y)\big)\Big). \end{aligned} \tag{1.2.4}$$

Combining (1.2.1) and (1.2.2) with (1.2.3) and (1.2.4), we finally obtain

$$F^X_{n,k}(x) = 1-[1-F(x)]^k \sum_{i=0}^{n-1} \frac{\big[-k\ln\big(1-F(x)\big)\big]^i}{i!}, \tag{1.2.5}$$

$$F^X_{m,n,k}(x,y) = \begin{cases} 1-[1-F(x)]^k \sum_{i=0}^{m-1} \frac{\left[-k\ln\left(1-F(x)\right)\right]^i}{i!} - [1-F(y)]^k \\ \times \sum_{i=0}^{n-m-1}\sum_{j=0}^{i} \frac{(-1)^j\left[-k\ln\left(1-F(x)\right)\right]^{m+j}\left[-k\ln\left(1-F(y)\right)\right]^{i-j}}{(m-1)!j!(i-j)!(m+j)}, & x<y, \\ 1-[1-F(y)]^k \sum_{i=0}^{n-1} \frac{\left[-k\ln\left(1-F(y)\right)\right]^i}{i!}, & y\le x. \end{cases} \tag{1.2.6}$$

Clearly, writing x and y instead of $F(x)$ and $F(y)$, respectively, in the right-hand sides of (1.2.5) and (1.2.6), we obtain the distribution functions $F^U_{n,k}$ and $F^U_{m,n,k}$ of kth records based on standard uniform sequence.

1.3 Variances and covariances of order statistics and kth records

We use the Hoeffding (1940) formula for the covariance

$$\mathbb{C}ov(X,Y) = \iint_{\mathbb{R}^2} [H(x,y) - F(x)G(y)]dxdy \tag{1.3.1}$$

of random variables X and Y with joint distribution function H and marginals F and G, respectively (for a simple proof, see Lehmann, 1966). Note that either of conditions $F(x) = 0$

and $G(y)=0$ implies $H(x,y)=0$. Similarly, when from $F(x)=1$ and $G(y)=1$ follows that $H(x,y)=G(y)$ and $H(x,y)=F(x)$, respectively. Therefore, we can rewrite (1.3.1) as

$$\mathbb{C}ov(X,Y) = \iint_{0<F(x),G(y)<1} [H(x,y)-F(x)G(y)]\,dxdy. \tag{1.3.2}$$

Using (1.3.1), we also obtain

$$\begin{aligned}\mathbb{V}ar\,X &= \mathbb{C}ov(X,X) = \iint_{\mathbb{R}^2} [F(\min\{x,y\}) - F(x)F(y)]dxdy \\ &= 2\iint_{0<F(x)\le F(y)<1} F(x)[1-F(y)]\,dxdy. \end{aligned} \tag{1.3.3}$$

Note that each $F^U_{k:n}$ and $F^U_{n,k}$ is strictly increasing transformation of $[0,1]$ onto $[0,1]$. Therefore

$$\begin{aligned}\{0<F(x)\le F(y)<1\} &= \{0<F^U_{k:n}\big(F(x)\big)\le F^U_{k:n}\big(F(y)\big)<1\} \\ &= \{0<F^U_{n,k}\big(F(x)\big)\le F^U_{n,k}\big(F(y)\big)<1\}.\end{aligned}$$

By (1.1.1) and (1.3.3) we get

$$\begin{aligned}\mathbb{V}ar\,X_{k:n} &= 2\iint_{0<F(x)\le F(y)<1} F^U_{k:n}\big(F(x)\big)[1-F^U_{k:n}\big(F(y)\big)]\,dxdy \\ &= 2\iint_{0<F(x)\le F(y)<1} \left[\sum_{m=k}^{n} B_{m,n}\big(F(x)\big)\right]\left[\sum_{m=0}^{k-1} B_{m,n}\big(F(y)\big)\right] dxdy.\end{aligned}$$

Similarly, from (1.2.5) and (1.3.3) we conclude

$$\begin{aligned}\mathbb{V}ar\,R_{n,k} &= 2\iint_{0<F(x)\le F(y)<1} F^U_{n,k}\big(F(x)\big)[1-F^U_{n,k}\big(F(y)\big)]\,dxdy \\ &= 2\iint_{0<F(x)\le F(y)<1} \left[1-[1-F(x)]^k\sum_{i=0}^{n-1}\frac{\big[-k\ln\big(1-F(x)\big)\big]^i}{i!}\right] \\ &\times\ [1-F(y)]^k\sum_{i=0}^{n-1}\frac{\big[-k\ln\big(1-F(y)\big)\big]^i}{i!}\,dxdy\end{aligned}$$

In order to write down the covariances of order and record statistics in a relatively concise forms, we make the following observations. We represent $F^X_{k,m:n}(x,y)$ for $x<y$ as

$$F^X_{k,m:n}(x,y) = F^X_{k:n}(x) - \tilde{F}^X_{k,m:n}(x,y),$$

where

$$\begin{aligned}
\tilde{F}^X_{k,m:n}(x,y) &= \mathbb{P}(X_{k:n} \le x, X_{m:n} > y) \\
&= \mathbb{P}(\text{at least } k \text{ among } X_1,\dots,X_n \text{ are not greater than } x, \\
&\quad \text{and at least } n-m+1 \text{ among them are greater than } y) \\
&= \mathbb{P}(\text{at least } k \text{ among } X_1,\dots,X_n \text{ are not greater than } x, \\
&\quad \text{and at most } m-1 \text{ among them are not greater than } y) \\
&= \sum_{j=k}^{m-1}\sum_{i=k}^{j} \mathbb{P}(\text{exactly } j \text{ among } X_1,\dots,X_n \text{ are not greater than } y, \\
&\quad \text{and exactly } i \text{ among them are not greater than } x) \\
&= \sum_{j=k}^{m-1}\sum_{i=k}^{j} \binom{n}{i, j-i} F^i(x)(F(y)-F(x))^{j-i}(1-F(y))^{n-j} \\
&= \sum_{j=k}^{m-1}\sum_{i=k}^{j} B_{i,j,n}(F(x),F(y)) \\
&= \tilde{F}^U_{k,m:n}(F(x),F(y)), \qquad (1.3.4)
\end{aligned}$$

where

$$B_{i,j,n}(u,v) = \binom{n}{i, j-i} u^i(v-u)^{j-i}(1-v)^{n-j}, \qquad 0 < u \le v < 1,$$

for $0 \le i \le j \le n$ can be interpreted as is the probability that exactly i and j random variables among n i.i.d. standard uniform random variables are less than u and v, respectively. Similarly, we write

$$F^X_{m,n,k}(x,y) = F^X_{m,k}(x) - \tilde{F}^X_{m,n,k}(x,y),$$

with

$$\tilde{F}^X_{m,n,k}(x,y) = [1-F(y)]^k \sum_{i=0}^{n-m-1}\sum_{j=0}^{i} \frac{(-1)^j\big[-k\ln\big(1-F(x)\big)\big]^{m+j}\big[-k\ln\big(1-F(y)\big)\big]^{i-j}}{(m-1)!j!(i-j)!(m+j)}$$

(cf. (1.2.6)). Note further that $F^X_{k,m:n}(x,y) = F^U_{k,m:n}(F(x),F(y)) = 0$ iff either $F(x)=0$ or $F(y)=0$. Also, $F^X_{k,m:n}(x,y) = F^X_{k:n}(x)$ and $F^X_{k,m:n}(x,y) = F^X_{m:n}(y)$ under conditions $F(y)=1$ and $F(x)=1$, respectively. Analogous relations hold for functions $F^U_{m,n,k}$. Therefore we finally obtain

$$
\begin{aligned}
\mathbb{C}ov(X_{k:n}, X_{m:n}) = &\iint_{0<F(x)\leq F(y)<1} \Big[F^U_{k:n}\big(F(x)\big) - \tilde{F}^U_{k,m:n}\big(F(x), F(y)\big) \\
- &\ F^U_{k:n}\big(F(x)\big)F^U_{m:n}\big(F(y)\big) + F^U_{m:n}\big(F(x)\big) - F^U_{m:n}\big(F(x)\big)F^U_{k:n}\big(F(y)\big)\Big]dxdy \\
= &\iint_{0<F(x)\leq F(y)<1} \Big\{F^U_{k:n}\big(F(x)\big)\big[1 - F^U_{m:n}\big(F(y)\big)\big] \\
+ &\ F^U_{m:n}\big(F(x)\big)\big[1 - F^U_{k:n}\big(F(y)\big)\big] - \tilde{F}^U_{k,m:n}\big(F(x), F(y)\big)\Big\}dxdy \\
= &\iint_{0<F(x)\leq F(y)<1} \Bigg\{\left[\sum_{i=k}^{n} B_{i,n}\big(F(x)\big)\right]\left[\sum_{i=0}^{m-1} B_{i,n}\big(F(y)\big)\right] \\
+ &\left[\sum_{i=m}^{n} B_{i,n}\big(F(x)\big)\right]\left[\sum_{i=0}^{k-1} B_{i,n}\big(F(y)\big)\right] - \sum_{j=k}^{m-1}\sum_{i=k}^{j} B_{i,j,n}\big(F(x), F(y)\big)\Bigg\}dxdy \quad (1.3.5)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{C}ov(R_{m,k}, R_{n,k}) = &\iint_{0<F(x)\leq F(y)<1} \Big\{F^U_{m,k}\big(F(x)\big)\big[1 - F^U_{n,k}\big(F(y)\big)\big] \\
+ &\ F^U_{n,k}\big(F(x)\big)\big[1 - F^U_{m,k}\big(F(y)\big)\big] - \tilde{F}^U_{m,n,k}\big(F(x), F(y)\big)\Big\}dxdy \\
= &\iint_{0<F(x)\leq F(y)<1} \Bigg\{\left[1 - [1-F(x)]^k \sum_{i=0}^{m-1} \frac{\big[-k\ln\big(1-F(x)\big)\big]^i}{i!}\right] \\
\times &\ [1-F(y)]^k \sum_{i=0}^{n-1} \frac{\big[-k\ln\big(1-F(y)\big)\big]^i}{i!} \\
+ &\left[1 - [1-F(x)]^k \sum_{i=0}^{n-1} \frac{\big[-k\ln\big(1-F(x)\big)\big]^i}{i!}\right] \\
\times &\ [1-F(y)]^k \sum_{i=0}^{m-1} \frac{\big[-k\ln\big(1-F(y)\big)\big]^i}{i!} - [1-F(y)]^k \sum_{i=0}^{n-m-1}\sum_{j=0}^{i}(-1)^j \\
\times &\ \frac{\big[-k\ln\big(1-F(x)\big)\big]^{m+j}\big[-k\ln\big(1-F(y)\big)\big]^{i-j}}{(m-1)!j!(i-j)!(m+j)}\Bigg\}dxdy. \quad (1.3.6)
\end{aligned}
$$

1.4 Variation diminishing property

Variation diminishing property (VDP, for short) of a (finite or infinite) sequence of functions defined on a common interval domain asserts that arbitrary non-zero linear combination of the functions has not more sign changes than the respective sequence of combination coefficients. The property is one of crucial tools of our further analysis. The most popular VDP is famous Descartes's Rule of Signs, concerning power functions defined on positive half-axis.

Theorem 1 (see, e.g., Karlin and Studden, 1966, Corollary 1.4.4)**.** *Let* $p(x) = a_0 x^{b_0} + a_1 x^{b_1} + \cdots + a_n x^{b_n}$ *be a function with nonzero real coefficients* $a_0, \ldots, a_n$ *and real exponents* $b_0, \ldots, b_n$ *satisfying* $b_0 > b_1 > \ldots > b_n$*. Then* p *cannot have more positive roots (even counted with multiplicity) than the number of sign changes in the sequence* $a_0, \ldots, a_n$*.*

The theorem was stated without proof by Descartes in 1637 in the case when $b_1, \ldots, b_n$ were positive integers (see Descartes, 1954). A rigorous proof was delivered by Segner (1728). Here we present the general version with a short proof of Komornik (2006), affixed here for completeness.

Proof. Denote by $z(p)$ the number of positive roots of p and by $v(p)$ the number of sign changes in the sequence $a_0, \ldots, a_n$. We prove by induction on $v(p)$ that $z(p) \leq v(p)$. The case $v(p) = 0$ is obvious, since then all terms of $p(x)$ have the same sign for all $x > 0$, hence $z(p) = 0$.

If $v(p) > 0$, then we choose an index i for which $a_i a_{i+1} < 0$. Since dividing $p(x)$ by some power of x changes neither $v(p)$ nor $z(p)$, we may assume that $b_i > 0$ and $b_{i+1} < 0$. Let

$$p'(x) = \frac{d}{dx} p(x) = a'_0 x^{b_0 - 1} + a'_1 x^{b_1 - 1} + \cdots + a'_n x^{b_n - 1}.$$

Then $\mathrm{sgn}(a'_j) = \mathrm{sgn}(a_j)$ for all $j \in \{1, \ldots, i\}$ and $\mathrm{sgn}(a'_j) = -\mathrm{sgn}(a_j)$ for all $j \in \{i+1, \ldots, n\}$. Thus $v(p') = v(p) - 1$.

Now we show that $z(p') \geq z(p) - 1$. Let $x_1 < \ldots < x_k$ be all positive roots of $p(x)$ with respective multiplicities $m_1, \ldots, m_k$. Then these roots are also roots of $p'(x)$ with multiplicities $m_1 - 1, \ldots, m_k - 1$. Moreover by Rolle's theorem, each of the $k - 1$ open intervals (x_i, x_{i+1}) contains at least one root of $p'(x)$. Therefore

$$z(p') \geq (m_1 - 1) + \cdots + (m_k - 1) + k - 1 = m_1 + \cdots + m_k - 1 = z(p) - 1.$$

By inductive assumption we have $z(p) \leq z(p') + 1 \leq v(p') + 1 = v(p)$. □

It can be easily noted that the first and last sign of the combination $\sum_{i=0}^{n} a_i x^{b_i}$ is identical with the signs of the first and last non-zero coefficient of the combination. The above theorem immediately implies VDP of the Bernstein polynomials of a fixed degree in interval $(0, 1)$.

Lemma 1 (cf., e.g., Rychlik 2001, Lemma 14)**.** *The number of sign changes of a non-zero linear combination of Bernstein polynomials $\sum_{k=0}^{m} b_k B_{k,m}$ of degree m on the interval $(0,1)$ does not exceed the number of the sign changes of the sequence $(b_0,\dots,b_m)$. Moreover, the signs of the combination at the right neighborhood of 0 and the left neighborhood of 1 coincide with the signs of the first and last non-zero elements of the sequence, respectively.*

The first statement was proved in Schoenberg (1959). In fact, it simply follows from the representation

$$\sum_{k=0}^{m} b_k B_{k,m}(u) = \sum_{k=0}^{m} b_k \binom{m}{k} u^k (1-u)^{m-k} = (1-u)^m \sum_{k=0}^{m} b_k \binom{m}{k} x^k,$$

where $x = x(u) = \frac{u}{1-u}$ is a strictly increasing transformation of the unit interval onto $\mathbb{R}_+$. The latter claim is trivial.

The following lemma can also be easily deduced from the Theorem 1. Here we take the strictly increasing reversible function $x = x(u) = -\ln(1-u)$ that transforms $(0,1)$ onto $(0,+\infty)$. This implies that the VDP is inherited by the powers of functions $u \mapsto -\ln(1-u)$, $0 < u < 1$.

Lemma 2. *The number of sign changes of the linear combination*

$$\sum_{i=1}^{n} a_i[-\ln(1-u)]^{\alpha_i}, \qquad 0 < u < 1,$$

where $\sum_{i=1}^{n} |a_i| > 0$, and $-\infty < \alpha_1 < \ldots < \alpha_n < +\infty$, does not exceed the number of sign changes in the sequence $(a_1,\dots,a_n)$. Moreover, the signs of the function in the right vicinity of 0 and the left vicinity of 1 are identical with the signs of the first and last elements of $(a_1,\dots,a_n)$, respectively.

The extension the variation diminishing property to infinite sequences is proposed by Jasiński (2016, Proposition 2.1).

Lemma 3. *Consider a sequence of functions $(\varphi_i(x))_{i=1}^{\infty}$ defined on an interval $(a,b) \subset \mathbb{R}$. If $(\varphi_i(x))_{i=1}^{n}$ have the variation diminishing property for all $n = 1,2,\dots$, and sequence $(a_i)_{i=1}^{\infty}$, has $k < \infty$ sign changes, and*

$$g(x) = \sum_{i=1}^{\infty} a_i \varphi_i(x), \qquad a < x < b,$$

is well defined, then $g(x)$ has at most k sign changes.

This is proved by contradiction. If g has more than k sign changes, we choose $k+1$ arguments $a < x_1 < \ldots < x_{k+1} < b$ such that $g(x_i)g(x_{i+1}) < 0$, $i = 1, \ldots, k$. If follows that the same relations are preserved by finite sums $\sum_{i=1}^{n} a_i \varphi_i$ for sufficiently large n, and this contradicts VDP of $\sum_{i=1}^{n} a_i \varphi_i$.

The above lemma together with Theorem 1 imply the following.

Lemma 4. *Suppose that function $h : \mathbb{R}_+ \mapsto \mathbb{R}$ has an infinite Taylor expansion*

$$f(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}, \qquad x > 0,$$

where sequence $(a_i)_{i=0}^{\infty}$ changes the sign finitely many times. Then the number of sign changes of f in $\mathbb{R}_+$ does not exceed the number of sign changes of $(a_i)_{i=0}^{\infty}$. Moreover, the first sign of f coincides with the sign of the first non-zero element of $(a_i)_{i=0}^{\infty}$, whereas last one is identical with the ultimate sign of the sequence.

Chapter 2

Bounds on the expectations of L-statistics

Suppose that $X_1, \ldots, X_n$ are non-degenerate i.i.d. random variables with a finite mean $\mu = \mathbb{E}X_1$. The purpose of this chapter is to present sharp lower and upper bounds on the expectations of properly centered L-statistics $\mathbb{E}\sum_{i=1}^n c_i(X_{i:n} - \mu)$, with arbitrary $c_1, \ldots, c_n \in \mathbb{R}$ and their special cases, expressed in terms of the Gini mean difference scale units $\Delta = \mathbb{E}|X_1 - X_2|$. Centering is necessary in order to get non-trivial evaluations.

There is a vast literature devoted to inequalities for moments of order statistics, their functions and generalizations in various sampling models. The first result in the subject was due to Plackett (1947) who precisely estimated the expected sample range in the standard deviation units. Gumbel (1954) and Hartley and David (1954) independently provided analogous upper bounds for the sample maxima. Moriguti (1953) proposed a new evaluation technique based on the notion greatest convex minorant that is useful in getting sharp bounds for arbitrary L-statistics from general populations. In the paper, he presented algorithms for numerical calculations of the bounds for single order statistics and their differences. Balakrishnan (1993) developed the Moriguti method for analytic determination of bounds for several extreme order statistics. Arnold (1985) calculated tight upper bounds on the difference of expected sample maximum and population mean in scale units generated by central absolute population moments of various orders. Rychlik (1998) combined the methods of Moriguti (1953) and Arnold (1985) for presenting a method of calculating analogous inequalities for arbitrary L-statistics.

More precise bounds were derived for restricted families of parent distributions. Moriguti (1951) presented mean-standard deviation bounds for the sample maxima from symmetric populations, and extended the result to arbitrary order statistics in Moriguti (1953). Bounds for the maxima of symmetrically distributed populations in various scale units can be found in Arnold (1985), and similar results for arbitrary L-statistics are due to Rychlik (1998). Gajek and Rychlik (1998) proposed a new method of determining sharp bounds, based on the notion of projections, and used it for evaluating order statistics from symmetric unimodal distributions. Danielak (2003) applied the idea for obtaining analogous bounds in the families of distributions with decreasing densities and decreasing failure rates, whereas Goroncy and Rychlik (2015, 2016) solved a similar problem for the increasing density and increasing failure rate families, respectively. The projection method makes it possible to determine sharp positive upper bounds (and negative lower ones). Lower non-negative and upper non-positive bounds for arbitrary L-statistics from general populations expressed in various scale units based on central absolute moments were presented by Goroncy (2009). Rychlik (2009 a,b,c) derived similar evaluations for order statistics with small ranks coming from restricted classes of distributions. We finally mention evaluations of expected order statistics from the popular i.i.d. model of drawing with replacement from finite populations, due to Rychlik (2004). The result was extended by López-Blázquez and Rychlik (2008) to the case of arbitrary parent distributions on discrete populations of a fixed size.

2.1 General L-statistics

Before we formulate results, we introduce some auxiliary notions. Given $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$ with the arithmetic mean $\bar{c} = \frac{1}{n}\sum_{i=1}^n c_i$, we define vector $\mathbf{a} = \mathbf{a}(\mathbf{c}) = (a_0, \ldots, a_{n-2}) \in \mathbb{R}^{n-1}$ as follows

$$a_i = a_i(\mathbf{c}) = \frac{n(n-1)}{2(i+1)(n-i-1)} \sum_{k=1}^{i+1} (\bar{c} - c_k), \qquad i = 0, \ldots, n-2. \tag{2.1.1}$$

Furthermore, we put

$$\Xi_{\mathbf{c}}(u) = \sum_{i=0}^{n-2} a_i B_{i,n-2}(u), \qquad 0 \le u \le 1, \tag{2.1.2}$$

where $B_{k,m}$ are the Bernstein polynomials of degree m. Obviously, $\Xi_{\mathbf{c}}$ is a polynomial of degree $n-2$. Now we are in a position to state the main statement of this Chapter.

Theorem 2. *Assume that $X_1, \ldots, X_n$ are non-degenerate i.i.d. random variables with a finite mean $\mu = \mathbb{E}X_1$. Then, under the above notation, we have*

$$\min_{0 \le u \le 1} \Xi_{\mathbf{c}}(u) \le \mathbb{E}\, \frac{\sum_{i=1}^n c_i (X_{i:n} - \mu)}{\Delta} \le \max_{0 \le u \le 1} \Xi_{\mathbf{c}}(u).$$

If $0 < u_1 < \ldots < u_r < 1$ are all the inner points of the unit interval $[0,1]$, being the arguments of the maximum (minimum), then the upper (lower) bound is attained iff the *parent distribution function has the form*

$$F(x) = \begin{cases} 0, & x < x_0, \\ u_1, & x_0 \le x < x_1, \\ \vdots & \\ u_r, & x_{r-1} \le x < x_r, \\ 1, & x \ge x_r, \end{cases} \tag{2.1.3}$$

for arbitrary $x_0 \le \ldots \le x_r > x_0$.

If the maximum (minimum) amounts to $\Xi_{\mathbf{c}}(0)$*, then the upper (lower) bound is attained in the limit by any two-point distributions such that the probabilities of the smaller point tend to* 0*. Similarly, if the maximum (minimum) amounts to* $\Xi_{\mathbf{c}}(1)$*, the upper (lower) bound is attained in the limit by any two-point distributions such that the probabilities of the smaller point tend to* 1*.*

According to (2.1.3), any distribution function attaining the upper bound is discrete, and the set of its values that differ from 0 and 1 is a nonempty subset of $\{u_1, \ldots, u_r\}$. The number of jumps is greater by 1 than the cardinality of the subset. A polynomial of degree $n-2$ may have $\lfloor \frac{n}{2} \rfloor - 1$ local maxima at most. It is theoretically possible, but practically very unlikely that all the maxima belong to $(0,1)$ and provide identical values of the polynomial. For majority of L-statistics, especially these commonly used in statistical analysis, respective functions $\Xi_{\mathbf{c}}$ have either one or (quite rarely) two maxima in $(0,1)$. It also happens that the maximum is attained at either of the border points of the unit interval. Similar remarks concern the minima of $\Xi_{\mathbf{c}}$ for various $\mathbf{c}$.

Proof. We first get rid of μ in the representation of the expectation of centered L-statistics

$$\begin{aligned} \mathbb{E} \sum_{i=1}^{n} c_i (X_{i:n} - \mu) &= \mathbb{E} \sum_{i=1}^{n} c_i \left(X_{i:n} - \frac{1}{n} \sum_{k=1}^{n} X_{k:n} \right) \\ &= \mathbb{E} \left(\sum_{i=1}^{n} c_i X_{i:n} - \frac{1}{n} \sum_{i=1}^{n} c_i \sum_{k=1}^{n} X_{k:n} \right) = \mathbb{E} \sum_{i=1}^{n} (c_i - \bar{c}) X_{i:n}. \end{aligned}$$

Since the new coefficients $\tilde{c}_i = c_i - \bar{c}$, $i = 1, \ldots, n$, sum up to 0, we can represent the modified L-statistic $\sum_{i=1}^{n} \tilde{c}_i X_{i:n}$ as a linear combination of spacings

$$\sum_{i=1}^{n} \tilde{c}_i X_{i:n} = \sum_{i=1}^{n-1} b_i (X_{i+1:n} - X_{i:n}),$$

where

$$b_i = -\sum_{k=1}^{i} \tilde{c}_k = \sum_{k=1}^{i} (\bar{c} - c_k), \qquad i = 1, \ldots, n-1.$$

Now we use integral representations of the expected spacings

$$\mathbb{E}(X_{i+1:n} - X_{i:n}) = \int_{-\infty}^{\infty} B_{i,n}(F(x))\, dx, \qquad i = 1, \ldots, n-1,$$

due to Pearson (1902) (see also Jones and Balakrishnan, 2002, formula (3.1)). It is also useful in representing the Gini mean difference

$$\Delta = \mathbb{E}|X_1 - X_2| = \mathbb{E}(X_{2:2} - X_{1:2}) = \int_{-\infty}^{\infty} B_{1,2}(F(x))\, dx.$$

We also have

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^{n} c_i(X_{i:n} - \mu) &= \mathbb{E}\sum_{i=1}^{n-1} \left[\sum_{k=1}^{i} (\bar{c} - c_k)\right] (X_{i+1:n} - X_{i:n}) \\
&= \int_{-\infty}^{\infty} \sum_{i=1}^{n-1} \left[\sum_{k=1}^{i} (\bar{c} - c_k)\right] B_{i,n}(F(x))\, dx \\
&= \int_{-\infty}^{\infty} \sum_{i=1}^{n-1} \left[\sum_{k=1}^{i} (\bar{c} - c_k)\right] \frac{n(n-1)}{2i(n-i)} B_{i-1,n-2}(F(x)) B_{1,2}(F(x))\, dx \\
&= \int_{-\infty}^{\infty} \sum_{i=0}^{n-2} \frac{n(n-1)}{2(i+1)(n-i-1)} \left[\sum_{k=1}^{i+1} (\bar{c} - c_k)\right] B_{i,n-2}(F(x)) B_{1,2}(F(x))\, dx \\
&= \int_{-\infty}^{\infty} \Xi_{\mathbf{c}}(F(x)) B_{1,2}(F(x))\, dx.
\end{aligned}$$

For getting the upper bound we write

$$\mathbb{E}\sum_{i=1}^{n} c_i(X_{i:n} - \mu) \;\leq\; \sup_{-\infty < x < \infty} \Xi_{\mathbf{c}}(F(x)) \int_{-\infty}^{\infty} B_{1,2}(F(x))\, dx \leq \max_{0 \leq u = F(x) \leq 1} \Xi_{\mathbf{c}}(u)\, \Delta,$$

as desired.

We get the equality in the latter inequality if we do not exclude any $0 \leq u \leq 1$ from the possible values of the parent distribution function F. We also have the equality in the former one iff for almost all $x \in \mathbb{R}$ we have either $\Xi_{\mathbf{c}}(F(x)) = \max_{0 \leq u \leq 1} \Xi_{\mathbf{c}}(u)$ or $B_{1,2}(F(x)) = 0$. The latter is equivalent to either $F(x) = 0$ or $F(x) = 1$. The only possibility for attaining the bound is that the set of values of $F(x)$ is contained in $\{u_1,\dots,u_r\}\cup\{0,1\}$. By assumption, $\{F(x):\ x\in\mathbb{R}\}\cap\{u_1,\dots,u_r\}\neq\emptyset$.

Suppose now that $\Xi_{\mathbf{c}}(0)>\Xi_{\mathbf{c}}(u)$, $0<u<1$, and consider the family of parent distribution functions

$$F_u(x)=\begin{cases}0, & x<x_0,\\ u, & x_0\le x<x_1,\\ 1, & x\ge x_1,\end{cases}\qquad 0<u<1,$$

for some arbitrary $x_0<x_1$. Then

$$\mathbb{E}_u\sum_{i=1}^n c_i(X_{i:n}-\mu)=\Xi_{\mathbf{c}}(u)\,\Delta_u.$$

Letting $u\downarrow 0$, by continuity of $\Xi_{\mathbf{c}}$ we obtain

$$\lim_{u\downarrow 0}\frac{\mathbb{E}_u\sum_{i=1}^n c_i(X_{i:n}-\mu)}{\Delta_u}=\Xi_{\mathbf{c}}(0)=\max_{0\le u\le 1}\Xi_{\mathbf{c}}(u).$$

Similarly, in the case $\Xi_{\mathbf{c}}(1)>\Xi_{\mathbf{c}}(u)$, $0<u<1$, yields

$$\lim_{u\uparrow 1}\frac{\mathbb{E}_u\sum_{i=1}^n c_i(X_{i:n}-\mu)}{\Delta_u}=\Xi_{\mathbf{c}}(1)=\max_{0\le u\le 1}\Xi_{\mathbf{c}}(u).$$

The proofs for the lower bound as well as for its attainability are analogous. □

If $n=2$, we trivially obtain

$$\mathbb{E}[c_1(X_{1:2}-\mu)+c_2(X_{2:2}-\mu)]=\frac{c_2-c_1}{2}\mathbb{E}(X_{2:2}-X_{1:2})=\frac{c_2-c_1}{2}\,\Delta.$$

From now on, we exclude this case from further analysis. When $n\ge 3$, in order to determine extreme values of $\Xi_{\mathbf{c}}(u)$, $0\le u\le 1$, we need to analyze behavior of the derivative

$$\Xi'_{\mathbf{c}}(u)=\sum_{i=0}^{n-3}\alpha_i B_{i,n-3}(u), \tag{2.1.4}$$

where

$$\begin{aligned}\alpha_i &= \alpha_i(\mathbf{c})=(n-2)(a_{i+1}-a_i)=\frac{n(n-1)(n-2)}{2}\\ &\times \left[\frac{\sum_{k=1}^{i+2}(\bar{c}-c_k)}{(i+2)(n-i-2)}-\frac{\sum_{k=1}^{i+1}(\bar{c}-c_k)}{(i+1)(n-i-1)}\right],\qquad i=0,\dots,n-3.\end{aligned} \tag{2.1.5}$$

The following three remarks are useful in calculating bounds for specific L-statistics.

Remark 1. *We have* $\Xi_{\mathbf{c}}(u) = -\Xi_{\mathbf{c}'}(1-u)$, $0 \le u \le 1$, *for some* $\mathbf{a} = \mathbf{a}(\mathbf{c}) = (a_0, \ldots, a_{n-2})$ *and* $\mathbf{a}' = \mathbf{a}(\mathbf{c}') = (a'_0, \ldots, a'_{n-2})$ *iff* $a_i = -a'_{n-2-i}$, $i = 0, \ldots, n-2$, *i.e. when* $\sum_{k=i+2}^{n} c_k = \sum_{k=1}^{n-i-1} c'_i$, $i = 0, \ldots, n-2$, *or just simply* $c'_i = c_{n+1-i}$, $i = 1, \ldots, n$. *The condition implies that the upper (lower) evaluation for* $\sum_{i=1}^{n} c_i X_{i:n}$ *is identical with the negative of the lower (upper, respectively) evaluation for* $\sum_{i=1}^{n} c_{n+1-i} X_{i:n}$. *Examples of such pairs are the single* j*th smallest and greatest order statistics. In consequence, for every L-statistics with* $c_i = c_{n+1-i}$, $i = 1, \ldots, n$, *the lower bound is the negative of the upper one. It holds for the linear combinations of quasi-midranges* $\sum_{i=1}^{\lfloor n/2 \rfloor} c_i (X_{i:n} + X_{n+1-i:n})$ $(+c_{\frac{n+1}{2}} X_{\frac{n+1}{2}:n}$ *if* n *is odd).*

Remark 2. *We have* $\Xi_{\mathbf{c}}(u) = \Xi_{\mathbf{c}}(1-u)$, $0 \le u \le 1$, *iff* $a_i = a_{n-2-i}$, $i = 0, \ldots, n-2$, *which is consecutively equivalent to* $\frac{1}{n-2-2i} \sum_{k=i+2}^{n-1-i} c_k = \bar{c}$, $i = 0, \ldots, \lfloor \frac{n-3}{2} \rfloor$, *and* $\frac{1}{2}(c_i + c_{n+1-i}) = \bar{c}$ *for* $i = 1, \ldots, n$. *This is satisfied by linear combinations of quasi-ranges and the sample mean* $\sum_{i=1}^{\lfloor n/2 \rfloor} d_i (X_{n+1-i:n} - X_{i:n}) + d \sum_{i=1}^{n} X_{i:n}$. *Under the condition, the set of maximum (minimum) points of* $\Xi_{\mathbf{a}}(u)$, $0 \le u \le 1$, *is symmetric about* $\frac{1}{2}$. *In particular,* $\Xi_{\mathbf{c}}(u)$ *has an extreme at* $u = \frac{1}{2}$.

Remark 3. *Notice that the vector transformations* $\mathbf{a} : \mathbb{R}^n \mapsto \mathbb{R}^{n-1}$ *and* $\alpha : \mathbb{R}^n \mapsto \mathbb{R}^{n-2}$ *described by formulae (2.1.1) and (2.1.5), respectively, are linear. In consequence, functions (2.1.2) and (2.1.4) are linear operators acting on vectors of coefficients* $\mathbf{c} \in \mathbb{R}^n$. *The observation will be useful in our further calculations.*

2.2 Single order statistics

Obviously, $X_{r:n}$ is an L-statistic with the coefficient vector $\mathbf{c}(r)$ such that $c_i(r) = \delta_{ir}$. Simple calculations show that

$$a_i(\mathbf{c}(r)) = \begin{cases} \frac{n-1}{2(n-i-1)}, & \text{if} \quad i \le r-2, \\ -\frac{n-1}{2(i+1)}, & \text{if} \quad i \ge r-1. \end{cases}$$

It follows that the respective polynomials $\Xi_{r:n}(u) = \sum_{i=0}^{n-2} a_i(\mathbf{c}(r)) B_{i,n-2}(u)$ have particular forms:

$$\Xi_{1:n}(u) = -\sum_{i=0}^{n-2} \frac{n-1}{2(i+1)} B_{i,n-2}(u), \tag{2.2.1}$$

$$\Xi_{2:n}(u) = \frac{1}{2} B_{0,n-2}(u) - \sum_{i=1}^{n-2} \frac{n-1}{2(i+1)} B_{i,n-2}(u), \tag{2.2.2}$$

$$\Xi_{r:n}(u) = \sum_{i=0}^{r-2} \frac{n-1}{2(n-i-1)} B_{i,n-2}(u) - \sum_{i=r-1}^{n-2} \frac{n-1}{2(i+1)} B_{i,n-2}(u), \qquad 3 \le r \le n-2, \tag{2.2.3}$$

$$\Xi_{n-1:n}(u) = \sum_{i=0}^{n-3} \frac{n-1}{2(n-i-1)} B_{i,n-2}(u) - \frac{1}{2} B_{n-2,n-2}(u), \tag{2.2.4}$$

$$\Xi_{n:n}(u) = \sum_{i=0}^{n-2} \frac{n-1}{2(n-i-1)} B_{i,n-2}(u). \tag{2.2.5}$$

The corresponding derivatives are

$$\Xi'_{1:n}(u) = \sum_{i=0}^{n-3} \frac{(n-1)(n-2)}{2(i+1)(i+2)} B_{i,n-3}(u), \tag{2.2.6}$$

$$\Xi'_{2:n}(u) = -\frac{(n+1)(n-2)}{4} B_{0,n-3}(u) + \sum_{i=1}^{n-3} \frac{(n-1)(n-2)}{2(i+1)(i+2)} B_{i,n-3}(u), \tag{2.2.7}$$

$$\Xi'_{r:n}(u) = \sum_{i=0}^{r-3} \frac{(n-1)(n-2)}{2(n-i-2)(n-i-1)} B_{i,n-3}(u) - \frac{(n^2-1)(n-2)}{2r(n+1-r)} \times B_{r-2,n-3}(u) + \sum_{i=r-1}^{n-3} \frac{(n-1)(n-2)}{2(i+1)(i+2)} B_{i,n-3}(u), \qquad 3 \le r \le n-2, \tag{2.2.8}$$

$$\Xi'_{n-1:n}(u) = \sum_{i=0}^{n-4} \frac{(n-1)(n-2)}{2(n-i-2)(n-i-1)} B_{i,n-3}(u) - \frac{(n+1)(n-2)}{4} B_{n-3,n-3}(u), \tag{2.2.9}$$

$$\Xi'_{n:n}(u) = \sum_{i=0}^{n-3} \frac{(n-1)(n-2)}{2(n-i-2)(n-i-1)} B_{i,n-3}(u). \tag{2.2.10}$$

Proposition 1. *(i) For the extreme order statistics, we have*

$$-\frac{n-1}{2} \le \mathbb{E}\frac{X_{1:n}-\mu}{\Delta} \le -\frac{1}{2},$$
$$\frac{1}{2} \le \mathbb{E}\frac{X_{n:n}-\mu}{\Delta} \le \frac{n-1}{2}.$$

The lower and upper bounds in the above relations are attained in the limit by the two-point distributions such that the probabilities of the smaller points tend to 0 *and* 1*, respectively.*

(ii) The derivatives (2.2.7) and (2.2.9) have unique zeros $v_1(2)$ and $u_1(n-1) = 1 - v_1(2)$, respectively. Moreover,

$$\begin{aligned} \Xi_{2:n}(v_1(2)) &\le \mathbb{E}\frac{X_{2:n}-\mu}{\Delta} \le \frac{1}{2}, &(2.2.11)\\ -\frac{1}{2} &\le \mathbb{E}\frac{X_{n-1:n}-\mu}{\Delta} \le \Xi_{2:n}(u_1(n-1)). &(2.2.12)\end{aligned}$$

The lower bound in (2.2.11) and the upper bound in (2.2.12) are attained by the two-point distributions such that the probability of the smaller points are $v_1(2)$ and $u_1(n-1)$, respectively. The upper bound in (2.2.11) and the lower one in (2.2.12) are attained in the limit by the two-point distributions such that the probabilities of the smaller point tend to 0 and 1, respectively.

(iii) For $3 \le r \le n-2$, the derivative (2.2.8) has exactly two zeros $u_1(r) < v_1(r)$ in $(0,1)$. Moreover,

$$\Xi_{r:n}(v_1(r)) \le \mathbb{E}\frac{X_{r:n}-\mu}{\Delta} \le \Xi_{r:n}(u_1(r)),$$

and the lower and upper bounds are attained by the two-point distributions such that the probabilities of the smaller points amount to $v_1(r)$ and $u_1(r)$, respectively.

By Remark 1, we have $u_1(n+1-r) = 1 - v_1(r)$, $v_1(n+1-r) = 1 - u_1(r)$, and $\Xi_{n+1-r:n}(u_1(n+1-r)) = -\Xi_{r:n}(v_1(r))$, $\Xi_{n+1-r:n}(v_1(n+1-r)) = -\Xi_{r:n}(u_1(r))$.

Proof. Due to Theorem 2, it suffices to determine the arguments and corresponding values of particular functions (2.2.1)–(2.2.5) which provide their global extremes over the interval $[0,1]$. By Lemma 1, functions (2.2.6) and (2.2.10) are positive in $(0,1)$. Hence (2.2.1) and (2.2.5) are increasing in $[0,1]$. Their extreme values are

$$\begin{aligned} \Xi_{1:n}(0) &= a_0(\mathbf{c}(1)) = -\frac{n-1}{2},\\ \Xi_{1:n}(1) &= a_{n-2}(\mathbf{c}(1)) = -\frac{1}{2},\\ \Xi_{n:n}(0) &= a_0(\mathbf{c}(n)) = \frac{1}{2},\\ \Xi_{n:n}(1) &= a_{n-2}(\mathbf{c}(n)) = \frac{n-1}{2}.\end{aligned}$$

Observe that the bounds for the sample maxima could be also deduced from ones for the sample minima and *vice versa* with use of Remark 1.

By (2.2.7) and Lemma 1, function (2.2.2) is first decreasing and then increasing. Since $\Xi_{2:n}(0) = a_0(\mathbf{c}(2)) = \frac{1}{2}$, and $\Xi_{2:n}(1) = a_{n-2}(\mathbf{c}(2)) = -\frac{1}{2}$, we immediately conclude that the global maximum and minimum are $\Xi_{2:n}(0) = \frac{1}{2}$ and $\Xi_{2:n}(v_1(2)) < -\frac{1}{2}$, respectively, where $v_1(2)$ is the unique zero of (2.2.7) in $(0,1)$. The bounds for the case $r = n-1$ are derived in much the same way.

For $3 \le r \le n-2$, Lemma 1 and (2.2.8) assert that (2.2.3) is either consecutively increasing, decreasing and eventually increasing in $(0,1)$, or simply increasing in the whole interval. Relations $\Xi_{r:n}(0) = a_0(\mathbf{c}(r)) = \frac{1}{2}$ and $\Xi_{r:n}(1) = a_{n-2}(\mathbf{c}(r)) = -\frac{1}{2}$ contradict the latter possibility. Therefore (2.2.8) has two zeros in $(0,1)$. The first one $u_1(r)$ provides the global maximum of (2.2.3), and the other $v_1(r)$ gives the minimum. This ends the proof. $\square$

Table 2.1 presents numerical values of upper bounds for the single order statistics $X_{r:n}$, $3 \le r < n = 12$. Every cell contains two numbers: the bound value for respective r and n (printed in bold), and the probability of the smaller point of the two-point distribution which attains the bound (printed in the standard type). We adhere to the above conventions presenting numerical evaluations later on. We have not included cases $r = 1$, $r = 2$ and $r = n$, because then the bounds have simple forms and do not have to be determined numerically. As one can expect, the bound values increase in the consecutive columns, and decrease in the rows. The column increase accelerates, and the row decrease slows down. Analogous tendencies reveal the probability values. The conclusions would not change if we included case $r = 2$ into the analysis. The bound values amount to 0.5 then, and are attained in the limit as the probability of the smaller point tends to 0. Using the Table, we can deduce the lower bounds as well. The bound for the rth smallest order statistic $X_{r:n}$ is identical with the negative of the upper one for the rth greatest order statistic $X_{n+1-r:n}$. The attainability parameter is obtained by subtracting the respective table value for $X_{n+1-r:n}$ for one.

2.3 Differences of order statistics

We consider $X_{s:n} - X_{r:n}$ for $1 \le r < s \le n$. Here the vector of coefficients is $\mathbf{c}(r,s) = \mathbf{c}(s) - \mathbf{c}(r)$ with $\bar{c} = \bar{c}(r,s) = 0$. We specify the elements of (2.1.1) as follows

$$a_i(\mathbf{c}(r,s)) = a_i(\mathbf{c}(s)) - a_i(\mathbf{c}(r))$$
$$= \begin{cases} 0, & i = 0,\dots,r-2, \\ \frac{n(n-1)}{2(i+1)(n-i-1)}, & i = r-1,\dots,s-2, \\ 0, & i = s-1,\dots,n-2. \end{cases}$$

Note that the first and last zeros vanish if $r = 1$ and $s = n$, respectively.
Put

$$\Xi_{r,s:n}(u) = \Xi_{\mathbf{c}(r,s)}(u), \qquad 1 \le r < s \le n, \quad 0 < u < 1. \tag{2.3.1}$$

Table 2.1: Upper bounds on expectations of single order statistics $\frac{\mathbb{E}X_{r:n}-\mu}{\Delta}$ for $3 \le r < n \le 12$.

n \ r	3	4	5	6	7	8	9	10	11
4	**0.54167** 0.16667								
5	**0.51445** 0.05911	**0.65146** 0.40724							
6	**0.50683** 0.02789	**0.57139** 0.21986	**0.78008** 0.55347						
7	**0.50380** 0.01543	**0.54259** 0.13582	**0.64610** 0.35661	**0.91620** 0.64539					
8	**0.50233** 0.00945	**0.52853** 0.09178	**0.59411** 0.24833	**0.72850** 0.45912	**1.05613** 0.70730				
9	**0.50154** 0.00622	**0.52053** 0.06603	**0.56711** 0.18318	**0.65296** 0.34346	**0.81499** 0.53602	**1.19828** 0.75145			
10	**0.50107** 0.00432	**0.51550** 0.04974	**0.55087** 0.14110	**0.61254** 0.26741	**0.71579** 0.42077	**0.90392** 0.59498	**1.34184** 0.78436		
11	**0.50077** 0.00312	**0.51214** 0.03879	**0.54019** 0.11235	**0.58759** 0.21489	**0.66180** 0.34011	**0.78106** 0.48352	**0.99444** 0.64126	**1.48635** 0.80976	
12	**0.50058** 0.00233	**0.50976** 0.03110	**0.53272** 0.09181	**0.57080** 0.17709	**0.62798** 0.28163	**0.71343** 0.40183	**0.84793** 0.53493	**1.08606** 0.67841	**1.63154** 0.82992

Furthermore

$$
\begin{aligned}
&\alpha_i(\mathbf{c}(r,s)) = \alpha_i(\mathbf{c}(s)) - \alpha_i(\mathbf{c}(r)) \\
= &\begin{cases}
0, & i = 0,\dots,r-3, \\
(n-2)a_{i+1}(\mathbf{c}(r,s)), & i = r-2, \\
(n-2)[a_{i+1}(\mathbf{c}(r,s)) - a_i(\mathbf{c}(r,s))], & i = r-1,\dots,s-3, \\
-(n-2)a_i(\mathbf{c}(r,s)), & i = s-2, \\
0, & i = s-1,\dots,n-3.
\end{cases}
\end{aligned}
\tag{2.3.2}
$$

The first and last zeros disappear when $r \le 2$ and $s \ge n-1$, respectively. The second and fourth rows vanish for $r = 1$ and $s = n$, respectively. The third one does not appear for the spacings, i.e. when $s = r+1$. We also check that under condition $r - 1 \le i \le s - 3$ yields $a_{i+1}(\mathbf{c}(r,s)) < (=,>) a_i(\mathbf{c}(r,s))$ iff $i < (=,>) \frac{n-3}{2}$, respectively. This implies that all the elements of the third row are non-positive and non-negative iff $s \le \frac{n+3}{2}$ and $r \ge \frac{n-1}{2}$, respectively. They are first negative and ultimately positive when $r < \frac{n-1}{2}$ and $s > \frac{n+3}{2}$. Presenting below the bounds on the order statistics differences, we consider two cases separately. In Proposition 2, we analyze the possibilities that the difference contains at least one of the sample extremes. The remaining ones $2 \le r < s \le n-1$ are treated in Proposition 3.

Proposition 2. *(i) For the sample range we have*

$$2 - 2^{2-n} \le \mathbb{E}\frac{X_{n:n} - X_{1:n}}{\Delta} \le \frac{n}{2}.$$

The lower bound is attained by the symmetric two-point distributions. The upper one is attained in the limit by the two-point distributions such that the probability of one point decreases to 0.

(ii) If either $r = 1 < s < n$ or $1 < r < s = n$, then

$$0 \le \mathbb{E}\frac{X_{s:n} - X_{r:n}}{\Delta} \le \frac{n}{2}.$$

In the former (latter) case the lower and upper bounds are attained in the limit by the two-point distributions such that the probabilities of the smaller points tend to 1 *and* 0*, respectively (*0 *and* 1*, respectively).*

Proof. (i) By the comments following (2.3.2), function $\Xi_{1,n:n}$ is first decreasing and then increasing. Due to Remark 2, this is symmetric about $\frac{1}{2}$. Therefore

$$
\begin{aligned}
\max_{0 \le u \le 1} \Xi_{1,n:n}(u) &= \Xi_{1,n:n}(0) = \Xi_{1,n:n}(1) = \frac{n}{2}, \\
\min_{0 \le u \le 1} \Xi_{1,n:n}(u) &= \Xi_{1,n:n}\left(\frac{1}{2}\right) = 2 - 2^{2-n},
\end{aligned}
$$

because

$$\Xi_{1,n:n}(u) = \frac{\sum_{i=1}^{n-1} B_{i,n}(u)}{B_{1,2}(u)} = \frac{1-(1-u)^n - u^n}{2u(1-u)}.$$

(ii) Suppose that $r = 1$ and $s < n$. If $s \le \frac{n+3}{2}$, then function $\Xi_{1,s:n}$ is decreasing by the VDP of its derivative. We have $\Xi_{1,s:n}(0) = \frac{n}{2}$ which is the upper bound, and $\Xi_{1,s:n}(1) = 0$ which is the lower one. If $s > \frac{n+3}{2}$, then $\Xi_{1,s:n}$ is decreasing in some neighborhoods of 0 and 1, but it is also possible that it is increasing in some inner subinterval (v_0, u_0), say, of $(0,1)$. Then the right end-point of the subinterval is a possible candidate for the global maximum. Suppose that $\Xi_{1,s:n}(u_0) > \Xi_{1,s:n}(0) = \frac{n}{2}$. Therefore for some two-point distribution we have

$$\Xi_{1,s:n}(u_0) = \mathbb{E}\frac{X_{s:n} - X_{1:n}}{\Delta} \le \mathbb{E}\frac{X_{n:n} - X_{1:n}}{\Delta} \le \frac{n}{2},$$

which contradicts part (i) of the proposition. Moreover, point v_0 cannot provide the global minimum less than $\Xi_{1,s:n}(1) = 0$ either, because

$$\mathbb{E}\frac{X_{s:n} - X_{1:n}}{\Delta} = \Xi_{1,s:n}(v_0) < \Xi_{1,s:n}(1) = 0$$

is impossible for the expectation of non-negative random function $X_{s:n} - X_{1:n}$. The proof for $1 < r < s = n$ is similar. □

Proposition 3. *Assume that* $2 \le r < s \le n-1$.
(i) If either $r \ge \frac{n-1}{2}$ *or* $s \le \frac{n+3}{2}$, *polynomial*

$$\Xi'_{r,s:n}(u) = \sum_{i=0}^{n-3} \alpha_i(\mathbf{c}(r,s)) B_{i,n-3}(u),$$

$0 \le u \le 1$, *(see (2.3.2)) has a unique zero* $u_1 = u_1(r,s)$, *say, in* $(0,1)$. *Also,*

$$0 \le \mathbb{E}\frac{X_{s:n} - X_{r:n}}{\Delta} \le \Xi_{r,s:n}(u_1).$$

(ii) If $r < \frac{n-1}{2}$ *and* $s > \frac{n+3}{2}$, *the polynomial has either one zero* $u_1 \in (0,1)$ *or three zeros* $u_1 < v_1 < u_2$, *where* u_1 *and* u_1, u_2 *are local maxima of* $\Xi_{r,s:n}$ *in the former and latter cases, respectively. In these cases we have*

$$\begin{aligned} 0 &\le \mathbb{E}\frac{X_{s:n} - X_{r:n}}{\Delta} \le \Xi_{r,s:n}(u_1), \\ 0 &\le \mathbb{E}\frac{X_{s:n} - X_{r:n}}{\Delta} \le \max\{\Xi_{r,s:n}(u_1), \Xi_{r,s:n}(u_2)\}, \end{aligned}$$

respectively.

The lower bounds in the inequalities of statements (i) and (ii) are attained in the limit by the two-point distributions with the contributions of one point decreasing to 0*. If the maximum of* $\Xi_{r,s:n}(u)$ *,* $0 \le u \le 1$*, is attained at a single point* u_i*, then the upper bound is attained by the two-point distribution with the probability mass of the smaller point equal to* u_i*. If the global maximum is attained simultaneously at* u_1 *and* u_2*, then the bound is attained by the parent distribution functions of the form*

$$F(x) = \begin{cases} 0, & x < x_0, \\ u_1, & x_0 \le x < x_1, \\ u_2, & x_1 \le x < x_2, \\ 1, & x \ge x_2, \end{cases}$$

for arbitrary $x_0 \le x_1 \le x_2 > x_0$*.*

Existence of two global maxima is possible, e.g. for the quasi-ranges with $s = n+1-r$. Then $\Xi_{r,s:n}(u)$ is symmetric about $\frac{1}{2}$. Moreover, for r located relatively close to 1 and far from $\frac{n}{2}$, the polynomial is bimodal. Clearly, the lower and upper bounds for $X_{r:n} - X_{s:n}$, $r < s$, are the negatives of the upper and lower bounds for $X_{s:n} - X_{r:n}$.

Proof. Under the assumptions of point (i), the derivative $\Xi'_{r,s:n}(u)$ changes its sign once, from $+$ to $-$. Therefore $\Xi_{r,s:n}(u)$ is unimodal with the global maximum at the zero point of its derivative. The lower bound coincides with

$$\Xi_{r,s:n}(0) = \Xi_{r,s:n}(1) = a_0(\mathbf{c}(r,s)) = a_{n-2}(\mathbf{c}(r,s)) = 0.$$

In case (ii), Lemma 1 asserts that possible sequence of signs of $\Xi'_{r,s:n}$ are either $+-+-$ or $+-$. In the latter one, we can repeat the arguments of part (a). Otherwise, there are two local maxima in $(0,1)$, and both are appropriate candidates for the global one. Notice that the local minimum point v_1 located between them cannot beat the interval endpoints in the contest for the global minimum, because it is impossible that for any non-degenerate parent distribution

$$\mathbb{E}\frac{X_{s:n} - X_{r:n}}{\Delta} = \Xi_{r,s:n}(v_1) < 0 = \Xi_{r,s:n}(0) = \Xi_{r,s:n}(1).$$

□

Remark 4. *The upper bounds for the spacings* $(s = r+1)$ *and second spacings* $(s = r+2)$ *can be calculated analytically, because then equations* $\Xi'_{r,s:n}(u) = 0$ *can be simplified to linear and quadratic equations, respectively. Note that the solutions to the cases* $r = 1$ *and* $s = n$

were precisely described in Proposition 2(b). Otherwise we write

$$\begin{aligned}
\mathbb{E}\frac{X_{r+1:n}-X_{r:n}}{\Delta} &\leq \Xi_{r,r+1:n}\left(\frac{r-1}{n-2}\right) \\
&= \frac{1}{2}\binom{n}{r}\frac{(r-1)^{r-1}(n-r-1)^{n-r-1}}{(n-2)^{n-2}}, \\
\mathbb{E}\frac{X_{r+2:n}-X_{r:n}}{\Delta} &\leq \Xi_{r,r+2:n}(u_1(r,r+2)),
\end{aligned}$$

where

$$u_1(r,r+2) = \begin{cases} \frac{1}{2}, & \text{if } r=\frac{n-1}{2}, \\ \frac{n-3+2r(r-1)-\sqrt{a(r,n)}}{2(n-2)(2r-n-1)}, & \text{otherwise.} \end{cases}$$

and $a(r,n)=[n-3+2r(r-1)]^2-4(r^2-1)(n-2)(2r-n-1)$.

Remark 5. *The conclusions of Proposition 3(b) can be specified more precisely for the quasi-ranges, i.e. the differences of* r*th largest and smallest order statistics,* $2\leq r\leq\frac{n}{2}$*. By Remark 2, function* $\Xi_{r,n+1-r:n}$ *is symmetric about* $\frac{1}{2}$*, and* $\Xi'_{r,n+1-r:n}$ *is antisymmetric. Hence the latter vanishes at* $\frac{1}{2}$*. If*

$$\Xi''_{r,n+1-r:n}\left(\frac{1}{2}\right)=\frac{n!}{(n-4)!2^{n-4}}\sum_{i=r}^{n-r}\binom{n}{i}[(2i-n)^2-n+2]<0, \tag{2.3.3}$$

then $\Xi_{r,n+1-r:n}$ *has a local maximum at* $\frac{1}{2}$*. Since it is positive on* $(0,1)$*, and vanishes at 0 and 1, and has either one or three local extremes in* $(0,1)$*, the latter possibility is excluded, and* $\Xi_{r,n+1-r:n}\left(\frac{1}{2}\right)$ *is the global maximum. Summing up, the inequality in (2.3.3) implies*

$$\mathbb{E}\frac{X_{n+1-r:n}-X_{r:n}}{\Delta}\leq\Xi_{r,n+1-r:n}\left(\frac{1}{2}\right)=2^{1-n}\sum_{i=r}^{n-r}\binom{n}{i}, \tag{2.3.4}$$

which can be also written as $(2-2^{2-n})\sum_{i=0}^{r-1}\binom{n}{i}$*. The equality is attained by the symmetric two-point distributions.*

If the inequality in (2.3.3) is reversed, then $\Xi_{r,n+1-r:n}$ *has a local minimum at* $\frac{1}{2}$ *and two global maxima attained at two arguments* $0<u_1<\frac{1}{2}<u_2=1-u_1<1$ *located symmetrically about* $\frac{1}{2}$*. Accordingly,*

$$\mathbb{E}\frac{X_{n+1-r:n}-X_{r:n}}{\Delta}\leq\Xi_{r,n+1-r:n}\left(u_1\right)=\Xi_{r,n+1-r:n}\left(u_2\right),$$

where u_1 is the unique solution to

$$\begin{aligned}\frac{\Xi'_{r,n+1-r:n}(u)}{n(n-1)(n-2)} &= \frac{B_{r-2,n-3}(u) - B_{n-r-1,n-3}(u)}{r(n-r)} \\ &+ \sum_{i=r-2}^{n-r-2} \frac{(2i+3-n)B_{i,n-3}(u)}{(i+1)(i+2)(n-i-1)(n-i-2)} = 0\end{aligned}$$

in $\left(0, \frac{1}{2}\right)$. This bound is attained by the families of two-point distributions where the probability of one point is u_1, and symmetric three-point distributions where the probabilities of the extreme points are equal to u_1, and the middle point has probability $u_2 - u_1 = 1 - 2u_1$.

Note that for $\frac{n-\sqrt{n-2}}{2} \leq r \leq \frac{n}{2}$ we have $(2i-n)^2 < n-2$ for all $r \leq i \leq n-r$, and (2.3.4) follows. However, the assumption is very restrictive, and (2.3.4) holds true for much greater range of $2 \leq r \leq \frac{n}{2}$.

Table 2.2 contains numerical approximations of upper bounds for the differences of orders statistics $X_{s:n} - X_{r:n}$, $1 < r < s < n$ from the samples of size $n = 12$. Clearly, the bounds increase with respect to s and decrease with respect to r. The attainability parameters increase both in the columns and rows. The differences are hardly visible if r is small and s is large. However, if we added the extra row for $r = 1$ and column for $s = n = 12$, we would note a rapid rise of the bound value to $\frac{n}{2} = 6$. These bounds are attained in the limit by the two-valued distributions as the probability mass concentrates at one of them. It is seemingly surprising that in the same way we obtain the zero lower bounds for all the cases except for the sample range for which the strictly positive bound (here $2 - 2^{-10} \approx 1.99902$) is attained by the symmetric two-point distributions. On the other hand, the distributions provide the upper bounds for the rth quasi-ranges $X_{13-r:n} - X_{r:n}$ $r = 3, 4, 5, 6$. The bound for the second quasi-range $X_{11:12} - X_{2:12}$ is the unique among $X_{s:12} - X_{r:12}$, $1 \leq r < s \leq 12$, which was determined by maximizing a bimodal function (the other ones were unimodal). Since the function is symmetric as well its maximum was attained at two arguments. This implies that this bound is attained by three families of distributions: one symmetric three-point, and two two-valued. We also observe that the bound values are symmetric about the opposite diagonal of the Table, and the respective probability mass parameters sum up to 1. This is an implication of the analytic identity $\Xi_{r,s:n}(u) = \Xi_{n+1-s,n+1-r:n}(1-u)$.

2.4 Selected L-estimates

We start with studying trimmed means

$$T_r = \sum_{i=1}^{n} t_i X_{i:n} = \frac{1}{n+2-2r} \sum_{i=r}^{n+1-r} X_{i:n}, \qquad 2 \leq r \leq \frac{n}{2}, \tag{2.4.1}$$

Table 2.2: Upper bounds on expectations of differences of order statistics $\frac{\mathbb{E}X_{s:n}-X_{r:n}}{\Delta}$ for $1 < r < s < n = 12$.

r \ s	3	4	5	6	7	8	9	10	11
2	**1.27849** 0.1	**1.83618** 0.13403	**2.09697** 0.16299	**2.21157** 0.18594	**2.25382** 0.20092	**2.26531** 0.20753	**2.26740** 0.20921	**2.26766** 0.20947	**2.26768** 0.20949 0.79051
3		**0.73820** 0.2	**1.20639** 0.24177	**1.50697** 0.28167	**1.69613** 0.32080	**1.81078** 0.36049	**1.87784** 0.40450	**1.92285** 0.5	**2.26766** 0.79053
4			**0.55033** 0.3	**0.96871** 0.34582	**1.28553** 0.39203	**1.52411** 0.44130	**1.70801** 0.5	**1.87784** 0.59550	**2.26740** 0.79079
5				**0.47298** 0.4	**0.87858** 0.44871	**1.22461** 0.5	**1.52411** 0.55870	**1.81078** 0.63951	**2.26531** 0.79247
6					**0.45117** 0.5	**0.87858** 0.55129	**1.28553** 0.60797	**1.69613** 0.67920	**2.25382** 0.79908
7						**0.47298** 0.6	**0.96871** 0.65418	**1.50697** 0.71833	**2.21157** 0.81406
8							**0.55033** 0.7	**1.20639** 0.75823	**2.09697** 0.83701
9								**0.73820** 0.8	**1.83618** 0.86597
10									**1.27849** 0.9

and Winsorized means

$$W_r = \sum_{i=1}^{n} w_i X_{i:n} = \frac{r}{n} X_{r:n} + \frac{1}{n} \sum_{i=r+1}^{n-r} X_{i:n} + \frac{r}{n} X_{n+1-r:n}, \qquad 2 \le r \le \frac{n-1}{2},$$

which are popular location estimates. In order to omit analyzing numerous special cases, we decided to focus on the most popular central versions.

2.4.1 Trimmed means

In the case of trimmed means, we introduce polynomials

$$\Xi_{\mathbf{t}(r)}(u) = \sum_{i=0}^{n-2} a_i(\mathbf{t}(r)) B_{i,n-2}(u), \tag{2.4.2}$$

$$\Xi'_{\mathbf{t}(r)}(u) = \sum_{i=0}^{n-3} \alpha_i(\mathbf{t}(r)) B_{i,n-3}(u), \tag{2.4.3}$$

with coefficients (cf (2.1.1) and (2.1.5))

$$a_i(\mathbf{t}(r)) = \begin{cases} \frac{n-1}{2(n-i-1)}, & i = 0, \dots, r-2, \\ \frac{(n-1)(r-1)}{2(n+2-2r)} \frac{n-2i-2}{(i+1)(n-i-1)}, & i = r-1, \dots, n-r-1, \\ -\frac{n-1}{2(i+1)}, & i = n-r, \dots, n-2, \end{cases}$$

and

$$\begin{aligned} \alpha_i(\mathbf{t}(r)) &= (n-2)[a_{i+1}(\mathbf{t}(r)) - a_i(\mathbf{t}(r))] \\ &= \begin{cases} \frac{(n-1)(n-2)}{2(n-i-1)(n-i-2)}, & 0 \le i \le r-3, \\ \frac{(n-1)(n-2)}{2} \left[\frac{(r-1)(n-2r)}{(n+2-2r)r(n-r)} - \frac{1}{n-r+1} \right], & i = r-2, n-r-1, \\ -\frac{(n-1)(n-2)(r-1)}{4(n+2-2r)} \frac{4(i-\frac{n-3}{2})^2+n^2-1}{(i+1)(i+2)(n-i-1)(n-i-2)}, & r-1 \le i \le n-r-2, \\ \frac{(n-1)(n-2)}{2(i+1)(i+2)}, & n-r \le i \le n-3, \end{cases} \end{aligned} \tag{2.4.4}$$

specified by (2.4.1).

Proposition 4. *If $r = 2$, then*

$$-\frac{1}{2} \le \frac{\mathbb{E} T_r - \mu}{\Delta} \le \frac{1}{2}.$$

The lower and the upper bounds are attained in the limit by the two-point distributions such that the probability of the smaller point tends to 1 and 0, respectively.

If $r \geq 3$, then (2.4.3) has exactly two zeros $0 < u_1 = u_1(r) < \frac{1}{2} < v_1 = v_1(r) = 1 - u_1 < 1$, mutually symmetric about $\frac{1}{2}$, and then

$$-\Xi_{\mathbf{t}(r)}(u_1) = \Xi_{\mathbf{t}(r)}(v_1) \leq \frac{\mathbb{E}T_r - \mu}{\Delta} \leq \Xi_{\mathbf{t}(r)}(u_1).$$

The lower and the upper inequalities become equalities for the two-point distributions such that the probabilities of the greater and smaller points, respectively, amount to u_1.

It follows from the proof that for $r \geq 3$ the upper bounds are greater than $\frac{1}{2}$ and the lower ones are less than $-\frac{1}{2}$.

Proof. Due to Theorem 2, it suffices to find the global extremes of (2.4.2) over the interval $[0, 1]$. To this end, we consider its derivative (2.4.3). By Lemma 1, analysis of signs in (2.4.4) is helpful. For $r = 2$, the first and last rows do not appear. The second one

$$\alpha_0(\mathbf{t}(2)) = \frac{(n-1)(n-2)}{2}\left[\frac{n-4}{2(n-2)^2} - \frac{1}{n-1}\right] = -\frac{n^2 - 3n + 4}{4(n-2)}$$

is negative, because so is the discriminant of the quadratic numerator in the last fraction. The elements of the third row are negative for $n \geq 5$, and do not appear for $n = 4$. Therefore $\alpha(\mathbf{t}(2)) \in \mathbb{R}_-^{n-2}$, (2.4.3) is negative, and (2.4.2) is decreasing from $\Xi_{\mathbf{t}(r)}(0) = a_0(\mathbf{t}(r)) = \frac{1}{2}$ to $\Xi_{\mathbf{t}(r)}(1) = a_{n-2}(\mathbf{t}(r)) = -\frac{1}{2}$. It suffices for deducing the first statement of the Proposition.

If $r \geq 3$, then the elements of the first and last rows in (2.4.4) are positive, and those of the third one are negative for $n \geq 2r + 1$. It may happen, that $n = 2r$, then the elements of the second row in (2.4.4) are obviously negative. So we can conclude that the elements of (2.4.4) are first positive, then negative and eventually positive. The VDP implies that (2.4.2) is either increasing on the whole $(0, 1)$ or first increasing, then decreasing and finally increasing there. The first possibility is excluded by the relations

$$\begin{aligned}\Xi_{\mathbf{t}(r)}(0) &= a_0(\mathbf{t}(r)) = \frac{1}{2},\\ \Xi_{\mathbf{t}(r)}(1) &= a_{n-2}(\mathbf{t}(r)) = -\frac{1}{2}.\end{aligned}$$

So it increases from $\frac{1}{2}$ at 0 to the global maximum $\Xi_{\mathbf{t}(r)}(u_1) > \frac{1}{2}$, then decreases to the global minimum $\Xi_{\mathbf{t}(r)}(v_1) < -\frac{1}{2}$, and ultimately increases to $-\frac{1}{2}$ at 1. Symmetry of coefficients $t_i(r) = t_{n+1-i}(r)$, $i = 1, \ldots, n$, implies $\Xi_{\mathbf{t}(r)}(u) = -\Xi_{\mathbf{t}(r)}(1-u)$, $0 \leq u \leq 1$, and, in consequence, $u_1 = 1 - v_1$ with $\Xi_{\mathbf{t}(r)}(u_1) = -\Xi_{\mathbf{t}(r)}(v_1)$. This ends the proof. $\square$

2.4.2 Winsorized means

Evaluating bounds for the Winsorized means we study the following linear combinations of Bernstein polynomials

$$\Xi_{\mathbf{w}(r)}(u) = \sum_{i=0}^{n-2} a_i(\mathbf{w}(r))B_{i,n-2}(u), \tag{2.4.5}$$

$$\Xi'_{\mathbf{w}(r)}(u) = \sum_{i=0}^{n-3} \alpha_i(\mathbf{w}(r))B_{i,n-3}(u), \tag{2.4.6}$$

with the coefficients

$$a_i(\mathbf{w}(r)) = \begin{cases} \frac{n-1}{2(n-i-1)}, & i = 0, \ldots, r-2, \\ 0, & i = r-1, \ldots, n-r-1, \\ -\frac{n-1}{2(i+1)}, & i = n-r, \ldots, n-2, \end{cases}$$

and

$$\begin{aligned} \alpha_i(\mathbf{w}(r)) &= (n-2)[a_{i+1}(\mathbf{w}(r)) - a_i(\mathbf{w}(r))] \\ &= \begin{cases} \frac{(n-1)(n-2)}{2(n-i-1)(n-i-2)}, & i = 0, \ldots, r-3, \\ -\frac{(n-1)(n-2)}{2(n+1-r)}, & i = r-2, n-r-1, \\ 0, & i = r-1, \ldots, n-r-2, \\ \frac{(n-1)(n-2)}{2(i+1)(i+2)}, & i = n-r, \ldots, n-3, \end{cases} \end{aligned} \tag{2.4.7}$$

Proposition 5. *If $r = 2$, then*

$$-\frac{1}{2} \leq \frac{\mathbb{E}W_r - \mu}{\Delta} \leq \frac{1}{2},$$

and the lower and the upper bounds are attained in the limit by the two-point distributions such that the probability mass of the smaller one tends to 1 and 0, respectively.

Otherwise (2.4.6) has exactly two zeros $0 < u_1 < \frac{1}{2}$ and $v_1 = 1 - u_1$, and

$$-\Xi_{\mathbf{w}(r)}(u_1) = \Xi_{\mathbf{w}(r)}(v_1) \leq \frac{\mathbb{E}W_r - \mu}{\Delta} \leq \Xi_{\mathbf{w}(r)}(u_1) > \frac{1}{2}.$$

The equalities in the left- and right-hand side inequalities hold for the two-point distributions such that the probability of the greater and smaller point, respectively, is equal to u_1.

Proof. If $r = 2$, we can omit the first and last rows of (2.4.7) in our analysis. Vector $\alpha(\mathbf{w}(r))$ has two identical negative values at the ends, and zeros in between. It follows that (2.4.6) is negative and (2.4.5) is decreasing on $(0, 1)$. Its extreme values are

$$\Xi_{\mathbf{w}(r)}(0) = a_0(\mathbf{w}(r)) = \frac{1}{2}, \tag{2.4.8}$$

$$\Xi_{\mathbf{w}(r)}(1) = a_{n-2}(\mathbf{w}(r)) = -\frac{1}{2}. \tag{2.4.9}$$

For $r \geq 3$, the elements of sequence (2.4.7) are subsequently positive, negative, zero, again negative, and finally positive. It follows that (2.4.5) is either always increasing or subsequently increasing, decreasing, and increasing. Since (2.4.8) and (2.4.9) hold, the first option is false. Therefore (2.4.5) first increases to the global maximum at u_1, then decreases to the global minimum at v_1, and then increases to $-\frac{1}{2}$ at 1. Its symmetry causes that $v_1 = 1 - u_1$, and $\Xi_{\mathbf{w}(r)}(v_1) = -\Xi_{\mathbf{w}(r)}(u_1)$. □

2.4.3 Differences between trimmed and Winsorized means

Finally we evaluate the expected differences between the rth central trimmed and Winsorized means $T_r - W_r$, $2 \leq r \leq \frac{n-1}{2}$ in the Gini mean difference units. Here we use linearity of operators described in Remark 3.

Corollary 1. *For $2 \leq r \leq \frac{n-1}{2}$, polynomial $\Xi'_{\mathbf{t}(r)} - \Xi'_{\mathbf{w}(r)}$ has two zeros $0 < u_1 < \frac{1}{2}$ and $v_1 = 1 - u_1$, symmetric about $\frac{1}{2}$ and such that*

$$\Xi_{\mathbf{t}(r)}(v_1) - \Xi_{\mathbf{w}(r)}(v_1) \leq \frac{\mathbb{E}(T_r - W_r)}{\Delta} \leq \Xi_{\mathbf{t}(r)}(u_1) - \Xi_{\mathbf{w}(r)}(u_1),$$

and $\Xi_{\mathbf{t}(r)}(v_1) - \Xi_{\mathbf{w}(r)}(v_1) = \Xi_{\mathbf{w}(r)}(u_1) - \Xi_{\mathbf{t}(r)}(u_1)$. Moreover, the lower and upper bounds become equalities for the two-point distributions such that u_1 is the probability of the greater and smaller points, respectively.

Proof. Referring to Remark 3, we easily determine the coefficients of $\Xi_{\mathbf{t}(r)} - \Xi_{\mathbf{w}(r)}$ and its derivative in the Bernstein bases:

$$a_i(\mathbf{t}(r)) - a_i(\mathbf{w}(r)) = \begin{cases} 0, & i = 0, \dots, r-2, n-r, \dots, n-2, \\ \frac{(n-1)(r-1)}{2(n+2-2r)} \frac{n-2i-2}{(i+1)(n-i-1)}, & i = r-1, \dots, n-r-1, \end{cases}$$

and

$$\begin{aligned} &\alpha_i(\mathbf{t}(r)) - \alpha_i(\mathbf{w}(r)) \\ = &\begin{cases} 0, & i = 0, \dots, r-3, n-r, \dots, n-3, \\ \frac{(n-1)(n-2)(r-1)}{2(n+2-2r)} \frac{(n-2r)}{r(n-r)}, & i = r-2, n-r-1, \\ -\frac{(n-1)(n-2)(r-1)}{4(n+2-2r)} \frac{4(i-\frac{n-3}{2})^2+n^2-1}{(i+1)(i+2)(n-i-1)(n-i-2)}, & i = r-1, \dots, n-r-2. \end{cases} \end{aligned}$$

Table 2.3: Upper bounds on expectations of trimmed means $\frac{\mathbb{E}T_r-\mu}{\Delta}$, Winsorized means $\frac{\mathbb{E}W_r-\mu}{\Delta}$ and their differences $\frac{\mathbb{E}T_r-W_r}{\Delta}$ for $r=2\ldots,15$ and sample size $n=30$.

	T_r		W_r		T_r-W_r	
r	bound	u_1	bound	u_1	bound	u_1
2	**0.5**	0	**0.5**	0	**0.14073**	0.06012
3	**0.50084**	0.00344	**0.50032**	0.00128	**0.18472**	0.11091
4	**0.50682**	0.02151	**0.50358**	0.01108	**0.21072**	0.15652
5	**0.51701**	0.04765	**0.51011**	0.02794	**0.22783**	0.19822
6	**0.52995**	0.07743	**0.51918**	0.04923	**0.23850**	0.23652
7	**0.54495**	0.10880	**0.53038**	0.07360	**0.24315**	0.27157
8	**0.56162**	0.14060	**0.54355**	0.10029	**0.24116**	0.30329
9	**0.57969**	0.17202	**0.55866**	0.12884	**0.23110**	0.33137
10	**0.59887**	0.20238	**0.57578**	0.15894	**0.21082**	0.35525
11	**0.61881**	0.23104	**0.59506**	0.19038	**0.17795**	0.37431
12	**0.63897**	0.25724	**0.61671**	0.22300	**0.13967**	0.38837
13	**0.65839**	0.27992	**0.64095**	0.25649	**0.07768**	0.39793
14	**0.67536**	0.29762	**0.66694**	0.28827	**0.02766**	0.40377
15	**0.68733**	0.30885				

The latter consists of a series of zeros, a positive value, a series of negative ones, again the positive one, and zeros at the end. The zeros do not appear for $r=2$. Anyway, this implies that function $\Xi_{\mathbf{t}(r)}-\Xi_{\mathbf{w}(r)}$ is either increasing or increasing-decreasing-increasing. The former is impossible, because $\Xi_{\mathbf{t}(r)}(0)-\Xi_{\mathbf{w}(r)}(0)=a_0(\mathbf{t}(r))-a_0(\mathbf{w}(r))=\Xi_{\mathbf{t}(r)}(1)-\Xi_{\mathbf{w}(r)}(1)=a_{n-2}(\mathbf{t}(r))-a_{n-2}(\mathbf{w}(r))=0$. Consequently, the function has the global maximum at the first zero u_1 of its derivative, and the global minimum at the latter one $v_1=1-u_1$, owing to its symmetry. This completes the proof. □

In Table 2.3 we numerically compare the upper bounds on expectations of trimmed and Winsorized means and their differences for the samples of size $n=30$. They are accompanied by the parameters characterizing the two-point distributions attaining them. The lower bounds are the negatives of the upper ones, and their attainability parameters are symmetric with respect to $\frac{1}{2}$. We start with some intuitively clear observations. The bounds for the trimmed and Winsorized means increase as so does truncation level r. This happens because with increase of r these L-estimates more and more differ from the sample mean whose expectation is μ. The bounds for the differences T_r-W_r first increase and then decrease. For small r, statistics T_r and W_r do not differ much one from the other (and from the sample mean). Then the differences steadily increase till $r = 7 \approx \frac{n}{4}$ for which both T_r and W_r are constructed on the base of an approximately half of the sample. When r further increases, T_r and W_r approach one the other, and finally we get $T_{15} = W_{15}$ (this is the reason why we have not filled four cells in the last row of Table 2.3). However, we also observe the following. The upper bounds for the trimmed means are always greater than the other ones. The respective probabilities asserting attainability lie between those for differences and Winsorized means. These facts follow from our analytic considerations. The bounds are the maxima of certain positive polynomials over the interval $\left[0, \frac{1}{2}\right]$. The polynomials for T_r are the sums of those for W_r and $T_r - W_r$. So the bounds for T_r are greater than each of the latter ones and less than their sum. Mutual location of the characterizing parameters is also easily concluded from our proofs. The polynomials we consider in the study of bounds for W_r and $T_r - W_r$ are combinations of extreme and central Bernstein polynomials, respectively. These relations could not be simply deduced on the basis of statistical intuition only.

2.4.4 Mean absolute deviation from the median

Now we proceed to mean absolute deviation from the median

$$MAD(\mathbf{X}) = \frac{1}{n} \sum_{i=1}^{n} |X_i - med(\mathbf{X})|$$

which is a classic L-estimate of scale. Here

$$med(\mathbf{X}) = \begin{cases} X_{\frac{n+1}{2}:n}, & \text{if } n \text{ is odd,} \\ \frac{1}{2}\left(X_{\frac{n}{2}:n} + X_{\frac{n}{2}+1:n}\right), & \text{if } n \text{ is even,} \end{cases}$$

is the sample median so that we can write

$$MAD(\mathbf{X}) = \begin{cases} -\frac{1}{n}\sum_{i=1}^{\frac{n-1}{2}} X_{i:n} + \frac{1}{n}\sum_{i=\frac{n+3}{2}}^{n} X_{i:n}, & \text{if } n \text{ is odd,} \\ -\frac{1}{n}\sum_{i=1}^{\frac{n}{2}} X_{i:n} + \frac{1}{n}\sum_{i=\frac{n}{2}+1}^{n} X_{i:n}, & \text{if } n \text{ is even.} \end{cases}$$

We have the following.

Proposition 6. *For* $n \geq 3$

$$\frac{1}{2} \leq \frac{\mathbb{E}MAD(\mathbf{X})}{\Delta} \leq 1 - B_{\lfloor \frac{n}{2} \rfloor, n-1}\left(\frac{1}{2}\right).$$

The lower bound is attained in the limit for the two-point distributions such that the probability mass of one point decreases to zero. The upper one becomes equality for the symmetric two-point distributions.

The upper bound for each odd sample size $n = 2m+1$ coincides with that for the proceeding even size $n = 2m$. Using the Stirling approximation of the factorial sequence we note that the upper bounds tend to 1 at the rate $\sqrt{\frac{2}{\pi n}}$. Since the sample Gini mean difference

$$\hat{\Delta} = \frac{1}{n(n-1)} \sum_{i \neq j} |X_i - X_j| = \frac{2}{n(n-1)} \sum_{i=1}^{n} (2i-n-1) X_{i:n}$$

is an unbiased L-estimate of Δ, we can treat the result of Proposition 6 as a comparison of expectations of $MAD(\mathbf{X})$ and $\hat{\Delta}$. This shows that the former is always less. Moreover, the upper evaluation of the ratio depends on the sample size n of $MAD(\mathbf{X})$, and we can take any size m (greater or less than n) for calculating $\hat{\Delta}$. The lower one depends neither on n nor on m.

Proof. If n is odd, we immediately calculate

$$a_i = \begin{cases} \frac{n-1}{2(n-i-1)}, & i = 0, \dots, \frac{n-3}{2}, \\ \frac{n-1}{2(i+1)}, & i = \frac{n-1}{2}, \dots, n-2, \end{cases}$$

$$\alpha_i = \begin{cases} \frac{(n-1)(n-2)}{2(n-i-1)(n-i-2)}, & i = 0, \dots, \frac{n-5}{2}, \\ 0, & i = \frac{n-3}{2}, \\ -\frac{(n-1)(n-2)}{2(i+1)(i+2)}, & i = \frac{n-1}{2}, \dots, n-3. \end{cases}$$

By Lemma 1, $\Xi_{\mathbf{c}}(u)$ is first increasing and then decreasing in $(0,1)$. By Remark 2, it is symmetric as well, and so

$$\begin{aligned} \min_{0 \le u \le 1} \Xi_{\mathbf{c}}(u) &= \Xi_{\mathbf{c}}(0) = a_0 = \Xi_{\mathbf{c}}(1) = a_{n-2} = \frac{1}{2}, \\ \max_{0 \le u \le 1} \Xi_{\mathbf{c}}(u) &= \Xi_{\mathbf{c}}\left(\frac{1}{2}\right) = \sum_{i=0}^{\frac{n-3}{2}} \frac{n-1}{2(n-i-1)} B_{i,n-2}\left(\frac{1}{2}\right) + \sum_{i=\frac{n-1}{2}}^{n-2} \frac{n-1}{2(i+1)} B_{i,n-2}\left(\frac{1}{2}\right) \\ &= \sum_{i=0}^{\frac{n-3}{2}} B_{i,n-1}\left(\frac{1}{2}\right) + \sum_{i=\frac{n+1}{2}}^{n-1} B_{i,n-1}\left(\frac{1}{2}\right) = 1 - B_{\frac{n-1}{2},n-1}\left(\frac{1}{2}\right). \end{aligned}$$

For even n, the formulae are slightly modified

$$a_i = \begin{cases} \frac{n-1}{2(n-i-1)}, & i = 0, \dots, \frac{n}{2}-1, \\ \frac{n-1}{2(i+1)}, & i = \frac{n}{2}, \dots, n-2, \end{cases}$$

$$\alpha_i = \begin{cases} \frac{(n-1)(n-2)}{2(n-i-1)(n-i-2)}, & i = 0, \dots, \frac{n}{2}-2, \\ -\frac{(n-1)(n-2)}{2(i+1)(i+2)}, & i = \frac{n}{2}-1, \dots, n-3, \end{cases}$$

Table 2.4: Upper bounds on expectations of MAD for various sample sizes n.

n	2, 3	4, 5	6, 7	8, 9	10, 11	50, 51	100, 101	200, 201
bound	**0.5**	**0.625**	**0.6875**	**0.72656**	**0.75391**	**0.88772**	**0.92041**	**0.94365**

but the arguments remain the same, and lead to the conclusions

$$\begin{aligned}
\min_{0\le u\le 1} \Xi_{\mathbf{c}}(u) &= \Xi_{\mathbf{c}}(0) = a_0 = \Xi_{\mathbf{c}}(1) = a_{n-2} = \frac{1}{2}, \\
\max_{0\le u\le 1} \Xi_{\mathbf{c}}(u) &= \Xi_{\mathbf{c}}\left(\frac{1}{2}\right) = \sum_{i=0}^{\frac{n}{2}-1} \frac{n-1}{2(n-i-1)} B_{i,n-2}\left(\frac{1}{2}\right) + \sum_{i=\frac{n}{2}}^{n-2} \frac{n-1}{2(i+1)} B_{i,n-2}\left(\frac{1}{2}\right) \\
&= \sum_{i=0}^{\frac{n}{2}-1} B_{i,n-1}\left(\frac{1}{2}\right) + \sum_{i=\frac{n}{2}+1}^{n-1} B_{i,n-1}\left(\frac{1}{2}\right) = 1 - B_{\frac{n}{2},n-1}\left(\frac{1}{2}\right).
\end{aligned}$$

□

We present several exemplary values of upper bounds for MAD for various sample sizes in Table 2.4.

Chapter 3

Bounds on the variances of L-statistics

We consider i.i.d. random variables $X_1, \ldots, X_n$ with a positive and finite variance. Under the assumptions, we determine upper bounds on the ratios $\mathbb{V}ar\left(\sum_{i=1}^{n} c_i X_{i:n}\right)/\mathbb{V}ar\, X_1$ for arbitrary coefficients $c_1, \ldots, c_n \in \mathbb{R}$ of the combination. Then we present sufficient conditions assuring sharpness of the bounds. Further we determine assumptions under which the lower bounds on the variance ratios amount to zero. We also describe the families of two-point distributions which attain the bounds, possibly in the limit. In the sequel, we recall results of Papadatos (1995) who determined sharp upper and lower bounds on the variances of single order statistics. Then we provide analogous results for spacings $S_{i:n} = X_{i+1:n} - X_{i:n}$, $i = 1, \ldots, n-1$. Finally, we determine tight upper bounds for variances of arbitrary linear combinations of spacings based on three observations.

There are quite few research papers about bounds on variances of order statistics. In that mentioned above, Papadatos (1995) determined sharp lower and upper bounds on variances of single order statistics, expressed in terms of the single observation variance units. The upper bound for the special case of sample median was earlier presented in Yang (1982), and its tightness was proved by Lin and Huang (1989). Papadatos (1997) refined these results for the families of symmetric parent distributions. More precise solution to the problem was presented in Jasiński and Rychlik (2013). Much earlier, lower and upper bounds for the variances of sample extremes were delivered by Moriguti (1951). By now, there were not known respective evaluations for variances of combinations of two and more order statistics.

3.1 General L-statistics

For arbitrary non-zero $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$, we define

$$\begin{aligned}\Phi_{\mathbf{c}}(u,v) &= \left[\sum_{i=0}^{n-1} \frac{n}{i+1}\left(\sum_{k=1}^{i+1} c_k\right) B_{i,n-1}(u)\right]\left[\sum_{j=0}^{n-1} \frac{n}{n-j}\left(\sum_{m=j+1}^{n} c_m\right) B_{j,n-1}(v)\right] \\ &- \sum_{i=0}^{n-2}\sum_{j=i}^{n-2} \frac{n(n-1)}{(i+1)(n-1-j)}\left(\sum_{k=1}^{i+1} c_k\right)\left(\sum_{m=j+2}^{n} c_m\right) B_{i,j,n-2}(u,v). \qquad (3.1.1)\end{aligned}$$

Since

$$B_{i,j,n}(u,u) = \begin{cases} 0, & i < j, \\ B_{i,n}(u), & i = j, \end{cases}$$

we have

$$\begin{aligned}\Psi_{\mathbf{c}}(u) &= \Phi_{\mathbf{c}}(u,u) = \left[\sum_{i=0}^{n-1} \frac{n}{i+1}\left(\sum_{k=1}^{i+1} c_k\right) B_{i,n-1}(u)\right]\left[\sum_{j=0}^{n-1} \frac{n}{n-j}\left(\sum_{m=j+1}^{n} c_m\right) B_{j,n-1}(u)\right] \\ &- \sum_{i=0}^{n-2} \frac{n(n-1)}{(i+1)(n-1-i)}\left(\sum_{k=1}^{i+1} c_k\right)\left(\sum_{m=i+2}^{n} c_m\right) B_{i,n-2}(u). \qquad (3.1.2)\end{aligned}$$

Theorem 3. *Let $X_1, \ldots, X_n$ be i.i.d. with a finite and positive variance. Then for arbitrarily fixed non-zero $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$, we have*

$$\frac{\mathbb{V}ar(\sum_{i=1}^{n} c_i X_{i:n})}{\mathbb{V}ar\, X_1} \leq \sup_{0<u\leq v<1} \Phi_{\mathbf{c}}(u,v). \qquad (3.1.3)$$

If

$$\sup_{0<u\leq v<1} \Phi_{\mathbf{c}}(u,v) = \sup_{0<u<1} \Psi_{\mathbf{c}}(u), \qquad (3.1.4)$$

then bound (3.1.3) is sharp. Moreover, under notation

$$F_u(x) = \begin{cases} 0, & x < a, \\ u, & a \leq x < b, \\ 1, & x \geq b, \end{cases} \qquad (3.1.5)$$

the following yields.
(i) If $\sup_{0<u<1} \Psi_{\mathbf{c}}(u) = \Psi_{\mathbf{c}}(u_0)$ for some $0 < u_0 < 1$, then the bound is attained when the parent distribution function is F_{u_0}.
(ii) If $\sup_{0<u<1} \Psi_{\mathbf{c}}(u) = \Psi_{\mathbf{c}}(0)$, then (3.1.3) becomes equality in the limit for the parent

distribution functions (3.1.5) with $u \searrow 0$.
(iii) If $\sup_{0<u<1} \Psi_{\mathbf{c}}(u) = \Psi_{\mathbf{c}}(1)$, *then bound (3.1.3) is attained in the limit for the parent two-point distribution functions (3.1.5) with* $u \nearrow 1$.

Proof. By (1.1.1), (1.3.3), (1.3.4) and (1.3.5), with F denoting the parent distribution function, we have

$$\begin{aligned}
\mathbb{V}ar(\sum_{i=1}^{n} c_i X_{i:n}) &= \sum_{k=1}^{n} c_k^2\, \mathbb{V}ar\, X_{k:n} + 2\sum_{k<m} c_k c_m\, \mathbb{C}ov(X_{k:n}, X_{m:n}) \\
&= 2\iint_{0<F(x)\le F(y)<1} \Bigg\{ \sum_{k=1}^{n} c_k^2 F_{k:n}^U\big(F(x)\big)\big[1 - F_{k:n}^U\big(F(y)\big)\big] \\
&+ \sum_{k=1}^{n-1}\sum_{m=k+1}^{n} c_k c_m \Big[F_{k:n}^U\big(F(x)\big)\big[1 - F_{m:n}^U\big(F(y)\big)\big] \\
&+ F_{m:n}^U\big(F(x)\big)\big[1 - F_{k:n}^U\big(F(y)\big)\big] - \tilde{F}_{k,m:n}^U\big(F(x), F(y)\big)\Big]\Bigg\} dxdy \\
&= 2\iint_{0<F(x)\le F(y)<1} \Bigg\{ \Bigg[\sum_{k=1}^{n} c_k F_{k:n}^U\big(F(x)\big)\Bigg]\Bigg[\sum_{m=1}^{n} c_m[1 - F_{m:n}^U(F(y))]\Bigg] \\
&- \sum_{k=1}^{n-1}\sum_{m=k+1}^{n} c_k c_m \sum_{j=k}^{m-1}\sum_{i=k}^{j} B_{i,j,n}\big(F(x), F(y)\big)\Bigg\} dxdy \\
&= 2\iint_{0<F(x)\le F(y)<1} \Bigg\{ \Bigg[\sum_{i=1}^{n}\Bigg(\sum_{k=1}^{i} c_k\Bigg) B_{i,n}\big(F(x)\big)\Bigg] \\
&\times \Bigg[\sum_{j=0}^{n-1}\Bigg(\sum_{m=j+1}^{n} c_m\Bigg) B_{j,n}\big(F(y)\big)\Bigg] \\
&- \sum_{i=1}^{n-1}\sum_{j=i}^{n-1}\Bigg(\sum_{k=1}^{i} c_k\Bigg)\Bigg(\sum_{m=j+1}^{n} c_m\Bigg) B_{i,j,n}\big(F(x), F(y)\big)\Bigg\} dxdy \\
&= 2\iint_{0<F(x)\le F(y)<1} \Phi_{\mathbf{c}}\big(F(x), F(y)\big) F(x)[1 - F(y)]dxdy \\
&\le \max_{0\le u\le v\le 1} \Phi_{\mathbf{c}}(u, v)\mathbb{V}ar\, X_1,
\end{aligned}$$

which completes the proof of inequality (3.1.3).

The above calculations show that for the two-point distribution function (3.1.5), we get

$$\mathbb{V}ar_u\left(\sum_{i=1}^{n} c_i X_{i:n}\right) = \Psi_{\mathbf{c}}(u)\,\mathbb{V}ar_u\, X_1, \tag{3.1.6}$$

because u is the unique value of $F_u(x)$ and $F_u(y)$ except for 0 and 1. Accordingly, under (3.1.4) and condition (i) of Theorem 3, the upper bound is attained if the baseline distribution function is F_{u_0}. Attainability of the bound in cases (ii) and (iii) follows from (3.1.6) and continuity of (3.1.2) on $[0,1]$. □

Theorem 4. *Under assumptions of Theorem 3, if $c_1 c_n = 0$, then bound*

$$\frac{\mathbb{V}ar(\sum_{i=1}^{n} c_i X_{i:n})}{\mathbb{V}ar\, X_1} \geq 0$$

is sharp. If $c_1 = 0$ ($c_n = 0$, respectively), then the equality is attained under conditions of of Theorem 3(ii) ((iii), respectively).

Proof. We first note that $\Psi_{\mathbf{c}}(0) = nc_1^2$ and $\Psi_{\mathbf{c}}(1) = nc_n^2$. Therefore, if $c_1 = 0$, then under the procedure described in Theorem 3(ii) we obtain

$$\lim_{u \searrow 0} \frac{\mathbb{V}ar(\sum_{i=1}^{n} c_i X_{i:n})}{\mathbb{V}ar\, X_1} = \lim_{u \searrow 0} \Psi_{\mathbf{c}}(u) = \Psi_{\mathbf{c}}(0).$$

Similarly, we treat case $c_n = 0$. □

3.2 Single order statistics

Let $\mathbf{c}(k) = (0,\ldots,0,1,0,\ldots,0) \in \mathbb{R}^n$ be the vector with only non-zero element equal to 1 located at position $k \in \{1,\ldots,n\}$. Then obviously $\sum_{i=1}^{n} c_i X_{i:n} = X_{k:n}$, whereas function defined in (3.1.2) has the form

$$\Psi_{\mathbf{c}(k)}(u) = \left[\sum_{i=k}^{n} \binom{n}{i} u^{i-1}(1-u)^{n-i}\right]\left[\sum_{j=0}^{k-1} \binom{n}{j} u^{j}(1-u)^{n-j-1}\right], \quad 0 \leq u \leq 1. \tag{3.2.1}$$

Papadatos (1995) showed that for $1 < k < n$ yields

$$\frac{\mathbb{V}ar\, X_{k:n}}{\mathbb{V}ar\, X_1} \leq \max_{0 \leq u \leq 1} \Psi_{\mathbf{c}(k)}(u) = \Psi_{\mathbf{c}(k)}(u_0), \tag{3.2.2}$$

where $u_0 = u_0(k,n) \in (0,1)$ is the unique zero of the derivative of (3.2.1). In fact, he determined more precisely location of u_0 proving that $u_0 \in (\rho_1, \rho_2) \subset (0,1)$, where $\rho_1 = \rho_1(k,n)$ and $\rho_2 = \rho_2(k,n)$ are the only maxima of functions $[1-F^U_{k:n}(u)]/(1-u)$ and $F^U_{k:n}(u)/u$, respectively, in interval $(0,1)$. By Theorem 3(i), the equality in (3.2.2) holds for the two-point distribution functions (3.1.5) with $u = u_0$. For the case of extreme order statistics, Papadatos (1995) noted that

$$\begin{aligned}
\frac{\mathbb{V}ar\, X_{1:n}}{\mathbb{V}ar\, X_1} &\leq \max_{0 \leq u \leq 1} \Psi_{\mathbf{c}(1)}(u) = \Psi_{\mathbf{c}(1)}(0) = n, \\
\frac{\mathbb{V}ar\, X_{n:n}}{\mathbb{V}ar\, X_1} &\leq \max_{0 \leq u \leq 1} \Psi_{\mathbf{c}(n)}(u) = \Psi_{\mathbf{c}(n)}(1) = n.
\end{aligned}$$

Conditions of the above bounds attainability are described in Theorem 3(ii) and (iii), respectively.

Note also that for all $n \in \mathbb{N}$ we have

$$\begin{aligned}
\Psi_{\mathbf{c}(k)}(0) &= 0, \qquad k = 2, \dots, n, \\
\Psi_{\mathbf{c}(k)}(1) &= 0, \qquad k = 1, \dots, n-1.
\end{aligned}$$

Thus evaluation

$$\frac{\mathbb{V}ar\, X_{k:n}}{\mathbb{V}ar\, X_1} \geq 0$$

is tight for all $k \in \{1, \dots, n\}$ and $n \geq 2$. The equality is attained for $k = 1$ and $k = n$ under conditions of Theorem 3(iii) and (ii), respectively. For all other k, both procedures lead to the trivial zero bound.

3.3 Spacings

Spacings $S_{i:n} = X_{i+1:n} - X_{i:n}$ play important roles in various problems of statistical inference and other branches of applied probability. Comprehensive discussions of their properties and applications are presented, e.g., in Pyke (1965, 1972) and David and Nagaraja (2003). Various evaluations of the expectations of spacings were presented in the literature. The first ones are due to Moriguti (1953) who derived sharp bounds on expected spacings in the population standard deviation units. Raqab (2003) presented optimal upper bounds on the expectations of spacings in more general scale units, generated by central absolute populations moments of various orders $p \geq 1$. Danielak (2004) extended these results to arbitrary quasi ranges, i.e., differences of order statistics $X_{j:n} - X_{i:n}$, $1 \leq i < j \leq n$. Kozyra and Rychlik (2017a) obtained tight lower and upper bounds on the differences of expected order statistics measured in the Gini mean difference units. More stringent standard deviation bounds in the restricted families of decreasing density and decreasing failure rate distributions were determined by Danielak and Rychlik (2004). More general families of distributions with decreasing density and failure rate on the average were studied in Danielak and Rychlik (2003). Recently, Goroncy and Rychlik (2015,2016) presented analogous results for the distributions with increasing density and increasing failure rate functions, respectively. A first attempt of evaluating the expectations of spacings in finite populations was due to López-Blázquez (2000). Rychlik (2004) determined sharp upper bounds on the expectations of all quasi-ranges in the classic model of drawing with replacement. Similar results for the drawing without replacement scheme can be found in Papadatos and Rychlik (2004). Lower bounds for the spacings from the drawing with replacement model were presented by Goroncy and Rychlik (2009). Analogous results for the without replacement drawing scheme can be concluded from Goroncy and Rychlik (2008). All these bounds amount to zero except for the case $i = 1$, $n = 2$, for which we have $\mathbb{E}S_{1:2} \geq 2\left[\frac{N}{(N-1)^p+N-1}\right]^{1/p}(\mathbb{E}|X_1 - \mathbb{E}X_1|^p)^{1/p}$, where $p \geq 1$ and N denotes the population size.

This section is devoted to evaluations of variances of spacings. Put $\mathbf{c}(i, i+1) = \mathbf{c}(i+1) - \mathbf{c}(i) = (0, \dots, 0, -1, 1, 0, \dots, 0)$ for some $1 \leq i \leq n-1$ and $n \geq 2$. The vector has -1 and 1 at positions i and $i+1$, respectively, and zeros elsewhere. Then clearly $\sum_{j=1}^{n} c_j(i, i+1)X_{j:n} = X_{i+1:n} - X_{i:n} = S_{i:n}$, and function (3.1.1) can be written as

$$\Phi_{i:n}(u, v) = \Phi_{\mathbf{c}(i,i+1)}(u, v) = \binom{n}{i} u^{i-1}(1-v)^{n-i-1}\left[1 - \binom{n}{i} v^i (1-u)^{n-i}\right] \tag{3.3.1}$$

Here and later on we replace subscript $\mathbf{c}(i, i+1)$ by $i : n$ for convenience. We further obtain

$$\Psi_{i:n}(u) = \Phi_{i:n}(u, u) = \binom{n}{i} u^{i-1}(1-u)^{n-i-1}\left[1 - \binom{n}{i} u^i (1-u)^{n-i}\right]. \tag{3.3.2}$$

Then we have

$$\max_{0 \leq u \leq v \leq 1} \Phi_{i:n}(u, v) = \max_{0 \leq u \leq 1} \Psi_{i:n}(u),$$

because for fixed $u \in (0, 1)$ the function $\Phi_{i:n}(u, v)$, $u \leq v \leq 1$, is the product of two positive and decreasing functions of v. In consequence, we derive a straightforward conclusion of Theorems 3 and 4.

Proposition 7. *For arbitrary fixed $1 \leq i < n < \infty$, the bound*

$$\frac{\mathbb{V}ar\, S_{i:n}}{\mathbb{V}ar\, X_1} \leq \max_{0 \leq u \leq 1} \Psi_{i:n}(u) \tag{3.3.3}$$

is sharp. If $\max_{0 \leq u \leq 1} \Psi_{i:n}(u) = \Psi_{i:n}(u_0)$ for some $u_0 = u_0(i, n) \in (0, 1)$, then the upper bound in (3.3.3) is attained iff the parent distribution function is (3.1.5) with $u = u_0$. If

$\max_{0\le u\le 1}\Psi_{i:n}(u) = \Psi_{i:n}(0)$ *($\Psi_{i:n}(1)$, respectively), then this is attained in the limit by the parent distribution functions (3.1.5) with $u\downarrow 0$ ($u\uparrow 1$, respectively).*

For $1\le i<n\ge 3$, the trivial bound

$$\frac{\mathbb{V}ar\,S_{i:n}}{\mathbb{V}ar\,X_1}\ge 0$$

is sharp, and becomes equality in the limit for the parent distribution functions (3.1.5) with $u\downarrow 0$ when $i\ge 2$ and $u\uparrow 1$ when $i\le n-2$.

Remark 6. *Note that*

$$\Psi_{i:n}(u)=\binom{n}{i}u^{i-1}(1-u)^{n-i-1}\sum_{\substack{j=0\\ j\ne i}}^{n}B_{j,n}(u).$$

By Lemma 1, $\Psi_{i:n}(u)>0$ for all $n\ge 2$, $1\le i\le n-1$ and $u\in(0,1)$. Moreover, $\Psi_{i:n}(u)=0$ if either $i\ge 2$ and $u=0$ or $i\le n-2$ and $u=1$. This observation is intimately connected with the tight zero lower bound of Proposition 7 . Also, relation $\Psi_{i:n}(u)=\Psi_{n-i:n}(1-u)$ together with (3.3.3) imply that the upper bounds for the variances of $S_{i:n}$ and $S_{n-i:n}$ coincide. The same conclusion for the lower bounds results from the last claim of Proposition 7 .

In Lemmas 5 and 6, we precisely describe maxima of (3.3.2) for various parameters i and n.

Lemma 5. *For every $n\ge 3$*

(i) function $\Psi_{1:n}$ has a unique maximum at 0, and $\Psi_{1:n}(0)=n$,

(ii) function $\Psi_{n-1:n}$ has a unique maximum at 1, and $\Psi_{n-1:n}(1)=n$.

Proof. We first focus on the case $i=1$ and show that $\Psi_{1:n}$ is strictly decreasing on the interval $[0,1]$. Consider

$$\Psi'_{1:n}(u)=n(1-u)^{n-3}h_{1,n}(u),$$

where

$$h_{1,n}(u)=n(1-u)^{n-1}[2(n-1)u-1]-n+2.$$

Observe that $h_{1,n}(0)=-2(n-1)$, $h_{1,n}(1)=-(n-2)$ and

$$h'_{1,n}(u)=n(n-1)(1-u)^{n-2}(3-2nu),$$

which implies that $h_{1,n}$ is increasing on $\left[0, \frac{3}{2n}\right]$ and decreasing on $\left[\frac{3}{2n}, 1\right]$. We show that

$$h_{1,n}\left(\frac{3}{2n}\right) = 2n\left(1 - \frac{3}{2n}\right)^n - (n-2) < 0, \qquad n \geq 3,$$

which means that

$$\frac{2n-3}{n-2} < \left(\frac{2n}{2n-3}\right)^{n-1}, \qquad n \geq 3.$$

By the Bernoulli inequality,

$$\left(\frac{2n}{2n-3}\right)^{n-1} > 1 + \frac{3(n-1)}{2n-3} = \frac{5n-6}{2n-3}, \qquad n \geq 2.$$

It remains to notice that $\frac{5n-6}{2n-3} \geq \frac{2n-3}{n-2}$, which is equivalent to $(n-1)(n-3) \geq 0$, $n \geq 3$, and verifies desired claim. Summing up, we have $h_n(u) < 0$ and $\Psi'_{n:1}(u) < 0$ for all $0 < u < 1$ and $n \geq 3$, which implies that

$$\max_{u \in [0,1]} \Psi_{1:n}(u) = \Psi_{1:n}(0) = n, \qquad n \geq 3.$$

The conclusion for $i = n-1$ follows from the relation $\Psi_{i:n}(u) = \Psi_{n-i:n}(1-u)$ and the previous statement. □

Lemma 6. *Fix $n \geq 4$ and $2 \leq i \leq n-2$. Function (3.3.2) has either a unique local and global maximum or two local maxima and one local minimum between them. The local extreme arguments are the only zeros of the polynomial*

$$h_{i,n}(u) = [2(n-1)u - 2i + 1]B_{i,n}(u) - u(n-2) + i - 1. \tag{3.3.4}$$

Let $u_0 = u_0(i,n)$ denote the global maximum point.
Then $u_0(2,4) \in \left\{\frac{1}{2} - \frac{\sqrt{6}}{6} + \frac{\sqrt{3}}{6}, \frac{1}{2} + \frac{\sqrt{6}}{6} - \frac{\sqrt{3}}{6}\right\}$ and $\Psi_{2:4}(u_0(2,4)) = \frac{2\sqrt{2}}{3} \approx 0.94281$.
For $n > 4$ yields

(i) if $i < \frac{n}{2}$ ($i > \frac{n}{2}$), then $u_0(i,n) < \frac{1}{2}$ ($u_0(i,n) > \frac{1}{2}$, respectively),

(ii) if $n \geq 6$ is even, then $u_0\left(\frac{n}{2}, n\right) = \frac{1}{2}$ and

$$\Psi_{\frac{n}{2}:n}\left(\frac{1}{2}\right) = \binom{n}{\frac{n}{2}} \frac{1}{2^{n-2}} \left[1 - \binom{n}{\frac{n}{2}} \frac{1}{2^n}\right].$$

Proof. For given $n \geq 4$ and $2 \leq i \leq n-2$ we have:

$$\Psi'_{i:n}(u) = \binom{n}{i} u^{i-2}(1-u)^{n-2-i} h_{i,n}(u) = \binom{n}{i} \frac{u^{i-2}(1-u)^{n-2-i}}{n+1} \sum_{j=0}^{n+1} a_{j,n+1} B_{j,n+1}(u) \quad (3.3.5)$$

$\big($cf. (3.3.4)$\big)$, where

$$a_{j,n+1} = \begin{cases} -2(n-i)i, & \text{if } j = i, \\ 2(n-i)i, & \text{if } j = i+1, \\ (i-1)(n+1) - j(n-2), & \text{otherwise.} \end{cases} \quad (3.3.6)$$

Since $2 \leq i \leq n-2$, the arithmetic sequence $\tilde{a}_{j,n+1} = (i-1)(n+1) - j(n-2)$, $j \in \{0, \ldots n+1\}$, decreases from $\tilde{a}_{0,n+1} = (i-1)(n+1) > 0$ to $\tilde{a}_{n+1,n+1} = -(n+1)(n-1-i) < 0$. For any fixed $i \in \{2, \ldots, n-2\}$, if we replace any pair $\tilde{a}_{i,n+1}$, $\tilde{a}_{i+1,n+1}$ by arbitrary $a < 0$ and $b > 0$, we obtain another sequence with consecutive signs $+-+-$ (we suppressed here multiple pluses and minuses, and dropped a possible zero at $j = \frac{(i-1)(n+1)}{n-2}$). This holds true for (3.3.6), in particular. By Lemma 1, $\Psi_{i:n}$ is either first increasing and then decreasing or it is consecutively increasing, decreasing, increasing and ultimately decreasing.

We now treat the case $i = 2$, $n = 4$ with use of standard calculus tools. By (3.3.5),

$$\begin{aligned} \Psi'_{2:4}(u) &= (2u-1)[18u^2(1-u)^2 - 1] = (2u-1)[3\sqrt{2}u(1-u) - 1][3\sqrt{2}u(1-u) + 1] \\ &= 108 \left(u - \frac{1}{2} + \frac{\sqrt{6}}{6} + \frac{\sqrt{3}}{6}\right) \left(u - \frac{1}{2} + \frac{\sqrt{6}}{6} - \frac{\sqrt{3}}{6}\right) \left(u - \frac{1}{2}\right) \\ &\times \left(u - \frac{1}{2} - \frac{\sqrt{6}}{6} + \frac{\sqrt{3}}{6}\right) \left(u - \frac{1}{2} - \frac{\sqrt{6}}{6} - \frac{\sqrt{3}}{6}\right). \end{aligned}$$

Hence the derivative $\Psi'_{2:4}$ restricted to $[0,1]$ has three zeros at $\frac{1}{2}$, $\frac{1}{2} - \frac{\sqrt{6}}{6} + \frac{\sqrt{3}}{6}$, and $\frac{1}{2} + \frac{\sqrt{6}}{6} - \frac{\sqrt{3}}{6}$. Moreover $\Psi'_{2:4}(u) > 0$ iff either $u \in \left(0, \frac{1}{2} - \frac{\sqrt{6}}{6} + \frac{\sqrt{3}}{6}\right)$ or $u \in \left(\frac{1}{2}, \frac{1}{2} + \frac{\sqrt{6}}{6} - \frac{\sqrt{3}}{6}\right)$. By symmetry of the function about $\frac{1}{2}$, we get

$$\max_{u \in [0,1]} \Psi_{2:4}(u) = \Psi_{2:4}\left(\frac{1}{2} - \frac{\sqrt{6}}{6} + \frac{\sqrt{3}}{6}\right) = \Psi_{2:4}\left(\frac{1}{2} + \frac{\sqrt{6}}{6} - \frac{\sqrt{3}}{6}\right) = \frac{2\sqrt{2}}{3} \approx 0.94281.$$

(i) Now we proceed to $n \geq 5$. Observe that

$$\begin{aligned}
\Psi_{i:n}(u) &= \binom{n}{i} u^{i-1}(1-u)^{n-1-i} \left[\sum_{\substack{j=0 \\ i\neq j\neq n-i}}^{n} \binom{n}{j} u^j (1-u)^{n-j} + \binom{n}{i} u^{n-i}(1-u)^i \right] \\
&= \binom{n}{i} [u(1-u)]^{i-1}(1-u)^{n-2i} \sum_{\substack{j=0 \\ i\neq j\neq n-i}}^{n} \binom{n}{j} u^j (1-u)^{n-j} + \binom{n}{i}^2 [u(1-u)]^{n-1},
\end{aligned}$$

and

$$\Psi_{i:n}(1-u) = \binom{n}{i} [u(1-u)]^{i-1} u^{n-2i} \sum_{\substack{j=0 \\ i\neq j\neq n-i}}^{n} \binom{n}{j} u^j (1-u)^{n-j} + \binom{n}{i}^2 [u(1-u)]^{n-1}.$$

In consequence,

$$\Psi_{i:n}(u) - \Psi_{i:n}(1-u) = \binom{n}{i} [u(1-u)]^{i-1} \sum_{\substack{j=0 \\ i\neq j\neq n-i}}^{n} \binom{n}{j} u^j (1-u)^{n-j} [(1-u)^{n-2i} - u^{n-2i}].$$

The sign of the difference is identical with that of the expression in square brackets. Therefore for $i < \frac{n}{2}$ this difference is positive on $(0, \frac{1}{2})$ and negative on $(\frac{1}{2}, 1)$. If $i > \frac{n}{2}$, the signs are reversed. This immediately implies our claims.

(ii) Suppose finally that $n \geq 6$ is even and $i = \frac{n}{2}$. Due to (3.3.2), $\Psi_{\frac{n}{2}:n}$ is symmetric about $\frac{1}{2}$, and $\Psi_{\frac{n}{2}:n}(\frac{1}{2})$ is a local extreme. We prove that this is a maximum, verifying that $\Psi''_{\frac{n}{2}:n}(\frac{1}{2}) < 0$. Using $i = \frac{n}{2}$ for simplicity of notation we have

$$\begin{aligned}
\Psi''_{i:2i}(u) &= \frac{(2i)![u(1-u)]^{i-3}}{i!^4} \Big[i!^2 (i-1)(4iu^2 - 4iu - 6u^2 + i + 6u - 2) \\
&\quad - 2u^i(1-u)^i (2i)!(2i-1)(4iu^2 - 4iu - 3u^2 + i + 3u - 1) \Big], \\
\Psi''_{i:2i}\left(\frac{1}{2}\right) &= \frac{(2i)!}{2^{2i-1} i!^4} h(i),
\end{aligned}$$

where

$$h(i) = 4^{-i}(2i)!(2i-1) - i!^2(i-1)$$

determines the sign of $\Psi''_{i:2i}\left(\frac{1}{2}\right)$. We shall prove that $h(i) < 0$ for $i \geq 3$ by induction. We check that $h(3) = -\frac{63}{4}$ and assume that $h(i) < 0$ for some $i \geq 3$ which is equivalent to $\binom{2i}{i}4^{-i}\left(2+\frac{1}{i-1}\right) < 1$. We show that the relation holds for $i+1$ as well. Indeed,

$$\binom{2i+2}{i+1}4^{-i-1}\left(2+\frac{1}{i}\right) = \binom{2i}{i}4^{-i}\left(2+\frac{1}{i}\right)\frac{2i+1}{2(i+1)}$$
$$< \binom{2i}{i}4^{-i}\left(2+\frac{1}{i-1}\right)\frac{2i+1}{2i+2} < \binom{2i}{i}4^{-i}\left(2+\frac{1}{i-1}\right) < 1,$$

by the inductive assumption. This ends the proof. □

We are not able to arbitrate theoretically which functions $\Psi_{i:n}$, $2 \leq i \leq n-2$, $i \neq \frac{n}{2}$, have one and two local maxima. Also, in the latter case, we not have tools for deciding if both the local maxima are located in the same half of the unit interval. Numerical analysis of functions (3.3.2) for small n shows that two maxima appear only for $i = 2$, $n = 4$ (see Lemma 6). If n increases, the possibility of two maxima becomes less likely. Note that (3.3.2) can be represented as a linear combination of Bernstein polynomials $B_{j,2n-2}$, $j = i-1, \ldots, 2i-2, 2i, \ldots, n+i-1$, with positive coefficients. The full such combination with $j = i-1, \ldots, n+i-1$ amounts to $\binom{n}{i}u^{i-1}(1-u)^{n-i-1}$ is certainly unimodal. It seems that removing one component with $j = 2i-1$ does not violate the property, and becomes almost negligible, especially for large n.

Using Lemmas 5 and 6 we are able to specify general result of Proposition 7 for particular $1 \leq i \leq n-1$ and $n \geq 2$. Only case $i = 1$ and $n = 2$ described in Proposition 8 needs an additional justification. Propositions 9 and 10 are direct conclusions of Proposition 7 and Lemmas 5 and 6.

Proposition 8. *We have*

$$\frac{2}{3} \leq \frac{\mathbb{V}ar\, S_{1:2}}{\mathbb{V}ar\, X_1} \leq \Psi_{1:2}(0) = \Psi_{1:2}(1) = 2.$$

The lower inequality becomes equality iff X_1 and X_2 are uniformly distributed.

Writing here and later that $\frac{\mathbb{V}ar\, S_{i:n}}{\mathbb{V}ar\, X_1} \leq (\geq)\Psi_{i:n}(u_0)$, we mean that the upper (lower, respectively) bound amounts to $\Psi_{i:n}(u_0)$ and is attained by the two-point distribution (3.1.5) with $u = u_0$ if $0 < u_0 < 1$, and in the limit by a sequence of F_u with $u \to u_0$ if $u_0 = 0$ or $u_0 = 1$. We use the convention for the sake of brevity.

Proof. The upper bound is evident by Proposition 7 , since

$$\Psi_{1:2}(u) = 2[1-2u(1-u)] = 2-4u+4u^2, \qquad 0 \leq u \leq 1,$$

attains its maximal value 2 at 0 and 1. In order to establish the lower one, we first recall formula due to Irvin (1925)

$$\mathbb{E}\, S_{i:n}^2 = 2\binom{n}{i} \iint_{x\leq y} F^i(x)[1-F(y)]^{n-i} dx dy,$$

representing the second raw moments of spacings (see also Jones and Balakrishnan, 2002, formula (3.4)). This together with (1.3.3) imply that $\mathbb{E}\, S_{1:2}^2 = 2\, \mathbb{V}ar\, X_1$. Accordingly, the problem of minimizing

$$\frac{\mathbb{V}ar\, S_{1:2}}{\mathbb{V}ar\, X_1} = 2 - \frac{\mathbb{E}\, S_{1:2}^2}{\mathbb{V}ar\, X_1} \tag{3.3.7}$$

is dual to that of maximizing $\frac{|\mathbb{E}\, S_{1:2}|}{\sqrt{\mathbb{V}ar\, X_1}}$. We focus on the later one. Suppose that X_1, X_2 are independent, and have a common distribution function F with mean

$$\mu = \int_0^1 F^{-1}(x)\, dx,$$

and finite and positive variance

$$\sigma^2 = \int_0^1 [F^{-1}(x) - \mu]^2\, dx.$$

Then

$$\mathbb{E}\, S_{1:2} \;=\; \mathbb{E}[F^{-1}(U_{2:2}) - F^{-1}(U_{1:2})] = \int_{\mathbb{R}} [F^{-1}(x) - \mu][f_{2:2}(x) - f_{1:2}(x)] dx,$$

where $U_{1:2}$ and $U_{2:2}$ denote the minimum and maximum of two i.i.d. standard uniform random variables, and

$$f_{1:2}(x) = \begin{cases} 2(1-x), & \text{if } 0 < x < 1, \\ 0, & \text{otherwise,} \end{cases}$$
$$f_{2:2}(x) = \begin{cases} 2x, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise,} \end{cases}$$

stand for the respective density functions. By Cauchy-Schwarz inequality,

$$\begin{aligned} |\mathbb{E}\, S_{1:2}| &= 2\left|\int_0^1 [F^{-1}(x) - \mu](2x-1) dx\right| \\ &\leq 2\sqrt{\int_0^1 [F^{-1}(x) - \mu]^2 dx \int_0^1 (2x-1)^2 dx} = \frac{2\sqrt{3}}{3}\,\sigma. \end{aligned}$$

This is a special case of the classic bounds on the expectation of sample ranges due to Plackett (1947), and together with (3.3.7), determine the lower variance bound. Observe that the equality holds in the Cauchy-Schwarz inequality iff

$$F^{-1}(x) - \mu = \alpha(2x-1), \qquad 0 < x < 1, \tag{3.3.8}$$

for some real α. Since F^{-1} is nondecreasing and nonconstant function, α has to be positive. Condition $\int_0^1 [F^{-1}(x) - \mu]^2 dx = \sigma^2$ implies that $\alpha = \sqrt{3}\,\sigma$. Hence, equation (3.3.8) uniquely determines the quantile function of the uniform distribution on the interval $[\mu - \sqrt{3}\,\sigma, \mu + \sqrt{3}\,\sigma]$. Clearly, changing parameters μ and σ we obtain the uniform distribution on arbitrary intervals. These distributions attain the lower variance bound of Proposition 8. □

Proposition 9. *If $n \in \mathbb{N} \cap [3, \infty)$, then:*

$$\begin{aligned} 0 = \Psi_{1:n}(1) \le & \tfrac{\mathbb{V}ar\, S_{1:n}}{\mathbb{V}ar\, X_1} \le \Psi_{1:n}(0) = n, \\ 0 = \Psi_{n-1:n}(0) \le & \tfrac{\mathbb{V}ar\, S_{n-1:n}}{\mathbb{V}ar\, X_1} \le \Psi_{n-1:n}(1) = n. \end{aligned}$$

Proposition 10. *If $n \in \mathbb{N} \cap [4, \infty)$ and $i \in \{2, \ldots, n-2\}$, then*

$$0 = \Psi_{i:n}(0) = \Psi_{i:n}(1) \le \frac{Var\, S_{i:n}}{\mathbb{V}ar\, X_1} \le \Psi_{i:n}(u_0),$$

where u_0 is described in Lemma 6.
In particular, for even n and $i = \frac{n}{2}$, we have

$$\begin{aligned} 0 = \Psi_{2:4}(0) = \Psi_{2:4}(1) \le \frac{\mathbb{V}ar\, S_{2:4}}{\mathbb{V}ar\, X_1} & \le \Psi_{2:4}\left(\frac{1}{2} - \frac{\sqrt{6}}{6} + \frac{\sqrt{3}}{6}\right) \\ & = \Psi_{2:4}\left(\frac{1}{2} + \frac{\sqrt{6}}{6} - \frac{\sqrt{3}}{6}\right) = \frac{2\sqrt{2}}{3} \approx 0.94281, \\ 0 = \Psi_{\frac{n}{2}:n}(0) = \Psi_{\frac{n}{2}:n}(1) \le \frac{\mathbb{V}ar\, S_{\frac{n}{2}:n}}{\mathbb{V}ar\, X_1} & \le \Psi_{\frac{n}{2}:n}\left(\frac{1}{2}\right) = \binom{n}{\frac{n}{2}} \frac{1}{2^{n-2}} \left[1 - \binom{n}{\frac{n}{2}} \frac{1}{2^n}\right], \qquad n \ge 6. \end{aligned}$$

Table 3.1 presents numerical values of upper bounds $\Psi_{i:20}(u_0(i,20))$ on variances of spacings $S_{i:20}$ for samples of size $n = 20$ and $1 \le i \le 10$, together with respective arguments $u_0(i, 20)$ which describe the two-point distribution functions (3.1.5) attaining the bounds in (3.3.3). Respective values for $11 \le i \le 19$ are immediately deduced from the relations

Table 3.1: Upper bounds on variances of spacings $\frac{\mathbb{V}ar(X_{i+1:n}-X_{i:n})}{\mathbb{V}ar\,X_1}$ for $i=1,\dots,10$ and $n=20$.

i	$u_0(i,20)$	$\Psi_{i:20}(u_0(i,20))$	i	$u_0(i,20)$	$\Psi_{i:20}(u_0(i,20))$
1	0	**20**	6	0.27038	**0.75942**
2	0.04347	**3.25396**	7	0.32794	**0.67092**
3	0.09792	**1.71152**	8	0.38537	**0.61799**
4	0.15502	**1.17002**	9	0.44271	**0.58958**
5	0.21270	**0.90714**	10	0.5	**0.58061**

$u_0(i,n) = 1 - u_0(n-i,n)$ and $\Psi_{i:20}(u_0(i,n)) = \Psi_{n-i:n}(u_0(n-i,n))$. We can see that if i increases from 1 to 10, then $u_0(i,20)$ increase from 0 to 0.5, whereas $\Psi_{i:20}(u_0(i,20))$ decrease from 20 to 0.58061. From Proposition 10 and the Stirling formula we deduce that the upper bounds for the central spacings with $i=\frac{n}{2}$ decrease to 0 at the rate $4\sqrt{\frac{2\pi}{n}}$ as n increases to infinity. By Proposition 9, the respective bounds for the extreme spacings tend to infinity faster.

3.4 Linear combinations of spacings based on three observations

Proposition 11 presented below shows that determination of bounds on variances of L-statistics for general choices of combination coefficients is a difficult task.

Proposition 11. *If X_1, X_2, X_3 are i.i.d. with a finite and positive variance, then under notation $S_{i:3} = X_{i+1:3} - X_{i:3}$, $i=1,2$, for every $a_1, a_2 \in \mathbb{R}$ bound*

$$\frac{\mathbb{V}ar(a_1S_{1:3} + a_2S_{2:3})}{\mathbb{V}ar\,X_1} \le 3\max\{a_1^2, a_2^2\}.$$

is best possible. If $|a_1| \ge |a_2|$ ($|a_1| \le |a_2|$, respectively), then the equality is attained under conditions of Theorem 3(ii) $\big($(iii), respectively$\big)$.

Proof. Cases $a_1 = 0$ and $a_2 = 0$ were solved in Proposition 9. Since $\mathbb{V}ar(a_1S_{1:3} + a_2S_{2:3}) = a_1^2\mathbb{V}ar(S_{1:3}+aS_{2:3})$ for $a_1 \ne 0$ and $a = \frac{a_2}{a_1}$, it suffices to consider $\mathbb{V}ar(S_{1:3}+aS_{2:3})$ for arbitrary $a \in \mathbb{R}\setminus\{0\}$.

If $\mathbf{c} = (-1, 1-a, a)$, then (3.1.2) takes on the form

$$\begin{aligned}\Phi_a(u,v) &= (3-3v)[1-3v(1-u)^2] + 3a^2u[1-3v^2(1-u)] \\ &+ a[6v - 6u - 9(1-u)^2v^2 - 9u(1-u)v(1-v)].\end{aligned}$$

Note that this is a quadratic function in each u, v, and a. Our purpose is to calculate $\max_{0\le u\le v\le 1} \Phi_a(u,v)$.

We first show that we can exclude all the interior points of triangle $0 \le u \le v \le 1$. For fixed $a \ge 1$ and $0 < v < 1$, quadratic function $\Phi_a(u,v)$, $0 \le u \le v$, is convex, because

$$\frac{\partial^2}{\partial u^2}\Phi_a(u,v) = 18v(a-1)[v(a-1)+1] \ge 0.$$

Accordingly,

$$\Phi_a(u,v) \le \max\{\Phi_a(0,v), \Phi_a(v,v)\}, \qquad 0 \le u \le v.$$

Case $a < 1$ needs more elaborate arguments. Note that

$$\frac{\partial^2}{\partial v^2}\Phi_a(u,v) = 18(u-1)(a-1)[u(a-1)+1] \ge 0$$

for some $0 < u < 1$ iff either $0 < u < 1$ with $0 < a < 1$ or $0 < u \le \frac{1}{1-a}$ with $a < 0$. Under these restrictions

$$\Phi_a(u,v) \le \max\{\Phi_a(u,u), \Phi_a(u,1)\}, \qquad u \le v \le 1.$$

If $a < 0$ and $\frac{1}{1-a} < u < 1$, function $\Phi_a(u,\cdot)$ is increasing-decreasing, and has the global maximum at

$$v_0 = -\frac{3au^2 - 3au - 3u^2 + 2a + 6u - 4}{6(a-1)(u-1)[u(a-1)+1]}.$$

We show that $v_0 < u$ which gives $\Phi_a(u,v) \le \Phi_a(u,u)$ when $a < 0$ and $\frac{1}{1-a} < u < 1$ and $u \le v \le 1$. Relation $v_0 < u$ is equivalent to

$$(u-1)(a-1)[u(a-1)+1]g(a,u) > 0 \tag{3.4.1}$$

where

$$g(a,u) = a^2u^3 - a^2u^2 - 2au^3 + \frac{7}{2}au^2 + u^3 - \frac{3}{2}au - \frac{5}{2}u^2 + 2u + \frac{1}{3}a - \frac{2}{3}.$$

Under the assumptions, the first three factors in (3.4.1) are negative, and it suffices to check that $g(a,u) < 0$ then. Observe first that

$$g(0,u) = u^3 - \frac{5}{2}u^2 + 2u - \frac{2}{3} < 0$$

iff

$$u < \frac{\sqrt[3]{17+12\sqrt{2}}}{6} + \frac{1}{6\sqrt[3]{17+12\sqrt{2}}} + \frac{5}{6} \approx 1.4246$$

which implies that $g(0,u) < 0$ for all $0 < u < 1$. Moreover

$$\frac{\partial}{\partial a} g(a,u) = 2au^3 - 2au^2 - 2u^3 + \frac{7}{2}u^2 - \frac{3}{2}u + 1/3 > 0$$

when

$$a < \frac{12u^3 - 21u^2 + 9u - 2}{12u^2(u-1)}.$$

Since the right-hand side is positive for all $0 < u < 1$, the proof of inequality $v_0 < u$ is complete. In conclusion, under condition $a < 1$ we obtain

$$\Phi_a(u,v) \leq \max\{\Phi_a(u,u), \Phi_a(u,1)\}, \qquad u \leq v \leq 1.$$

Summing up, we proved that

$$\max_{0 \leq u \leq v \leq 1} \Phi_a(u,v) = \max_{0 \leq u \leq 1} \{\max\{\Phi_a(0,u), \Phi_a(u,1), \Phi_a(u,u)\}\}.$$

Now we exclude the last possibility proving that for all $0 \leq u \leq 1$

$$\Psi_a(u) \leq \Psi_a(0), \qquad \text{if } |a| \leq 1, \tag{3.4.2}$$

$$\Psi_a(u) \leq \Psi_a(1), \qquad \text{if } |a| \geq 1, \tag{3.4.3}$$

where $\Psi_a(u) = \Phi_a(u,u)$ (cf (3.1.2)). The first inequality is equivalent to

$$3u[(3u^3 - 3u^2 + 1)a^2 - 6u(u-1)^2 a - 9u^2 + 9u - 4] \leq 0.$$

The sign of the left-hand side is identical with that of the expression in the brackets. Since the coefficient $3u^3 - 3u^2 + 1$ associated with a^2 is positive for $0 \leq u \leq 1$, and so is the discriminant $4(6u^2 - 9u + 4)$ for all real u, the inequality holds true if

$$\frac{3u^3 - 6u^2 + 3u - \sqrt{6u^2 - 9u + 4}}{3u^3 - 3u^2 + 1} \leq a \leq \frac{3u^3 - 6u^2 + 3u + \sqrt{6u^2 - 9u + 4}}{3u^3 - 3u^2 + 1}.$$

The left- and right-hand side restrictions are for all $0 \leq u \leq 1$ less than -1 and greater than 1, respectively. Therefore relation (3.4.2) holds for all $|a| \leq 1$. Inequality (3.4.3) can be rewritten as

$$3(u-1)[(3u^3 + 1)a^2 + 6u^2(1-u)a + 3u^3 - 6u^2 + 3u - 1] \leq 0.$$

By similar arguments, this is true when either

$$a \leq \frac{3u^2(u-1) - \sqrt{6u^2 - 3u + 1}}{3u^3 + 1}$$

or

$$a \geq \frac{3u^2(u-1)+\sqrt{6u^2-3u+1}}{3u^3+1}.$$

Since the right-hand side expressions range over $[-1,1)$ and $[0,1]$, respectively, when $u \geq 0$, inequality (3.4.3) holds when $|a| \geq 1$ and $0 \leq u \leq 1$.

Now we can focus on

$$\begin{aligned}
\Phi_a(0,u) &= 3\,[3(1-a)u^2+2(a-2)u+1],\\
\Phi_a(u,1) &= 3a\,[3(a-1)u^2-2(a-2)u-1].
\end{aligned}$$

Simple calculations show that

$$\max_{0\leq u\leq 1}\Phi_a(0,u)=\begin{cases}\Phi_a(0,1)=-3a, & a\leq -1,\\ \Phi_a(0,0)=3, & -1\leq a\leq 2,\\ \Phi_a\left(0,\frac{a-2}{3(a-1)}\right)=\frac{a^2-a+1}{a-1}, & a\geq 2,\end{cases}$$

and

$$\max_{0\leq u\leq 1}\Phi_a(u,1)=\begin{cases}\Phi_a(1,1)=3a^2, & a\leq -1,\\ \Phi_a(0,1)=-3a, & -1\leq a\leq 0,\\ \Phi_a\left(\frac{a-2}{3(a-1)},1\right)=\frac{a(a^2-a+1)}{1-a}, & 0\leq a\leq \frac{1}{2},\\ \Phi_a(1,1)=3a^2, & a\geq \frac{1}{2}.\end{cases}$$

Summing up, we obtain

$$\max_{0\leq u\leq 1}\{\max\{\Phi_a(0,u),\Phi_a(u,1)\}\}=\begin{cases}\Phi_a(0,0)=3, & |a|\leq 1,\\ \Phi_a(1,1)=3a^2, & |a|\geq 1,\end{cases}$$

and

$$\frac{\mathbb{V}ar(a_1S_{1:3}+a_2S_{2:3})}{\mathbb{V}ar\,X_1}\leq\begin{cases}a_1^2\Psi_a(0)=3a_1^2, & |a_1|\geq|a_2|,\\ a_1^2\Psi_a(1)=3a_2^2, & |a_1|\leq|a_2|.\end{cases}$$

Since the extreme values of Φ_a appear at the vertices of the triangle hypotenuse, the variance bounds are sharp for all the pairs $a_1, a_2 \in \mathbb{R}$. They are attained under conditions of Theorem 3(ii) and (iii), when $|a_1| \geq |a_2|$ and $|a_1| \leq |a_2|$, respectively. □

Chapter 4

Bounds on expectations of linear combinations of kth records

Let $X_1, X_2, \ldots$ be i.i.d. random variables with common continuous cumulative distribution function F. In this chapter, we determine sharp lower and upper bounds for expectations of arbitrary linear combinations of respective kth records $\mathbb{E}\left[\sum_{i=1}^n c_i(R_{i,k}-\mu)\right]$, centered about the population mean $\mu = \mathbb{E}X_1$, and expressed in the Gini mean difference units $\Delta = \mathbb{E}|X_1 - X_2|$. Moreover, we specify the bounds on the expectations of centered kth records $R_{n,k}-\mu$, and their differences $R_{n,k}-R_{m,k}$.

By now, various evaluations of $\mathbb{E}\left[\sum_{i=1}^n c_i(R_{i,k}-\mu)\right]$ for specific $\mathbf{c} = (c_1, \ldots, c_n)$ were presented in terms of scale units $\sigma_p = \left[\mathbb{E}|X_1-\mu|^p\right]^{1/p}$ generated by pth absolute central moments. The first result of this type was presented by Nagaraja (1978) who applied the Schwarz inequality for getting sharp bounds on the expectations of the classic record values expressed in terms of the mean μ and standard deviation σ_2 of the parent distribution. Raqab (2000) used the Hölder inequality in order to receive bounds expressed in terms of other scale units σ_p, $p \geq 1$. He also derived refined estimates of the records coming from symmetric populations. Rychlik (1997) evaluated the expectations of record spacings $\mathbb{E}(R_{n,1}-R_{n-1,1})$ in the general populations as well as under the restrictions to the distributions with increasing density and increasing failure rate. Danielak (2005) generalized these results to arbitrary record increments $R_{n,1}-R_{m,1}$, $n > m$.

For general kth records, Grudzień and Szynal (1985) obtained non-optimal evaluations in terms of μ and σ_2 by direct use of the Schwarz inequality. Raqab (1997) applied a modification of the Schwarz inequality proposed by Moriguti (1953) in order to get optimal bounds. Raqab and Rychlik (2002) used both the Moriguti and Hölder inequalities and calculated the bounds measured in various σ_p units. Similar results for the differences of adjacent and non-adjacent kth records were derived by Raqab (2004), and Danielak and Raqab (2004a), respectively.

Goroncy and Rychlik (2011) determined the lower bounds on the expectations of centered values of kth records, and their differences expressed in σ_p units.

Raqab and Rychlik (2004) calculated optimal evaluations for the 2nd record values coming from symmetric populations. Gajek and Okolewski (2003) provided the sharp bounds on the expectations of kth records coming from the decreasing density and failure rate populations expressed in the population second raw moments. Optimal mean-variance inequalities for the expected kth record spacings from the above models were presented in Danielak and Raqab (2004b). Second record non-adjacent differences coming from populations with decreasing density functions were studied in Raqab (2007). Tight upper bounds for the kth record values from the decreasing generalized failure rate populations were established by Bieniek (2007). Klimczak (2007) calculated sharp bounds on the expectations of kth records and their differences coming from bounded populations. They were expressed in the scale units amounting to the lengths of the population support intervals.

4.1 Linear combinations of kth record values

By (1.2.5), for every $n, k \in \mathbb{N}$, the distribution function of the nth value of the kth record coming from the standard uniform distribution is following

$$F^U_{n,k}(u) \quad = \quad 1-(1-u)^k \sum_{i=0}^{n-1} \frac{[-k\ln(1-u)]^i}{i!}, \qquad 0<u<1, \tag{4.1.1}$$

whereas the composition $F^X_{n,k} = F^U_{n,k} \circ F$ is the distribution function of the nth value of kth record coming from the population with continuous distribution function F. Below we use the following notions

$$\xi_{n,k}(u) \quad = \quad (1-u)^{k-1} \sum_{i=0}^{n-1} \frac{\big[-k\ln(1-u)\big]^i}{i!} - 1, \tag{4.1.2}$$

$$\Xi_{n,k}(u) \quad = \quad \frac{\xi_{n,k}(u)}{2u}, \tag{4.1.3}$$

$$\begin{aligned}\xi_{\mathbf{c},k}(u) \quad &= \quad \sum_{i=1}^{n} c_i \xi_{i,k}(u) \\ &= \quad (1-u)^{k-1} \sum_{i=0}^{n-1} b_{i+1} \frac{[-k\ln(1-u)]^i}{i!} - b_1, \end{aligned}\tag{4.1.4}$$

$$\Xi_{\mathbf{c},k}(u) \quad = \quad \sum_{i=1}^{n} c_i \Xi_{i,k}(u) = \frac{\xi_{\mathbf{c},k}(u)}{2u}, \tag{4.1.5}$$

where $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{R}^n$, and $b_i = \sum_{j=i}^n c_j$, $i = 1, \dots, n$.

Theorem 5. *Let $X_1, X_2, \dots$ be an i.i.d. sequence with a common continuous distribution function, expectation $\mu = \mathbb{E}X_1 \in \mathbb{R}$, and Gini mean difference $\Delta = \mathbb{E}|X_1 - X_2|$. Let $R_{1,k}, R_{2,k}, \dots$ denote the respective sequence of kth upper records, and assume that $\mathbb{E}R_{n,k} < \infty$. Then for arbitrary $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{R}^n$, with the notation (4.1.2)-(4.1.5), we have*

$$\inf_{0<u<1} \Xi_{\mathbf{c},k}(u) \le \frac{\mathbb{E}\left[\sum_{i=1}^n c_i(R_{i,k} - \mu)\right]}{\Delta} \le \sup_{0<u<1} \Xi_{\mathbf{c},k}(u). \tag{4.1.6}$$

Let $F_{m,a}$ denote the distribution function of the uniform random variable on the interval $\left[a - \frac{1}{m}, a\right]$. If the supremum (infimum) in (4.1.5) is attained at some $0 < u_1 < 1$, then the upper (lower) bound in (4.1.6) is attained in the limit by the sequence of parent distribution functions $F_m = u_1 F_{m,a} + (1 - u_1)F_{m,b}$ for arbitrary $a < b$. If the supremum (infimum) is attained there in the limit as $u \searrow 0$ ($u \nearrow 1$), then the upper (lower) bound is attained in the limit by any sequence of distribution functions $F_m = u_m F_{m,a} + (1 - u_m)F_{m,b}$ as $m \to \infty$ and $u_m \searrow 0$ ($u_m \nearrow 1$, respectively) whereas $a < b$.

Proof. We start with a useful representation of the expectations of record spacings. For $1 \le i \le n-1$, we have

$$\begin{aligned} \mathbb{E}(R_{i+1,k} - R_{i,k}) &= \int_{-\infty}^{\infty} x\, F^U_{i+1,k}(F(dx)) - \int_{-\infty}^{\infty} x\, F^U_{i,k}(F(dx)) \\ &= \int_{-\infty}^{\infty} x \left((F^U_{i+1,k} - F^U_{i,k}) \circ F \right) (dx). \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned} \mathbb{E}(R_{i+1,k} - R_{i,k}) &= x\left[F^U_{i+1,k}\left(F(x)\right) - F^U_{i,k}\left(F(x)\right)\right]\Big|_{-\infty}^{\infty} \\ &- \int_{-\infty}^{\infty} \left(F^U_{i+1,k}\left(F(x)\right) - F^U_{i,k}\left(F(x)\right)\right) dx. \end{aligned}$$

Since $\mathbb{E}(|R_{i,k}|) < \infty$, $i = 0, \dots, n$, the first ingredient of the above sum is equal to 0 (note that for $x \nearrow \infty$, the difference of distribution functions can be treated as the negative of the difference of respective survival functions). Thus, by (4.1.1), we have

$$\mathbb{E}(R_{i+1,k} - R_{i,k}) = \int_{-\infty}^{\infty} \left[1 - F(x)\right]^k \frac{\left[-k\ln(1 - F(x))\right]^i}{i!} dx. \tag{4.1.7}$$

We also note that $R_{1,k} = X_{1:k}$ and $\mu = \mathbb{E}\big(\frac{1}{k}\sum_{j=1}^k X_j\big) = \mathbb{E}\big(\frac{1}{k}\sum_{j=1}^k X_{j:k}\big)$. Therefore

$$\begin{aligned}
\mathbb{E}(R_{n,k}-\mu) &= \mathbb{E}\left[\sum_{i=1}^{n-1}\big(R_{i+1,k}-R_{i,k}\big) - \frac{1}{k}\sum_{j=1}^{k}(X_{j:k}-X_{1:k})\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{n-1}\big(R_{i+1,k}-R_{i,k}\big) - \frac{1}{k}\sum_{j=2}^{k}\sum_{l=1}^{j-1}(X_{l+1:k}-X_{l:k})\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{n-1}\big(R_{i+1,k}-R_{i,k}\big) - \sum_{l=1}^{k-1}\frac{k-l}{k}(X_{l+1:k}-X_{l:k})\right].
\end{aligned}$$

We further use integral representations of the expected spacings

$$\mathbb{E}(X_{l+1:k}-X_{l:k}) = \int_{-\infty}^{\infty}\binom{k}{l}F^l(x)\big[1-F(x)\big]^{k-l}dx, \qquad l=1,\ldots,k-1, \tag{4.1.8}$$

due to Pearson (1902) (see also Jones and Balakrishnan, 2002, formula (3.1)). In particular, we have

$$\Delta = \mathbb{E}|X_1-X_2| = \mathbb{E}(X_{2:2}-X_{1:2}) = 2\int_{-\infty}^{\infty}F(x)[1-F(x)]dx. \tag{4.1.9}$$

Combining (4.1.7) and (4.1.8), we write

$$\begin{aligned}
\mathbb{E}(R_{n,k}-\mu) &= \int_{-\infty}^{\infty}\Bigg\{\big[1-F(x)\big]^k\sum_{i=1}^{n-1}\frac{[-k\ln(1-F(x))]^i}{i!} \\
&- \sum_{i=1}^{k-1}\frac{k-i}{k}\binom{k}{i}F^i(x)\big[1-F(x)\big]^{k-i}\Bigg\}dx \\
&= \int_{-\infty}^{\infty}2F(x)\big[1-F(x)\big]\Xi_{n,k}\big(F(x)\big)dx,
\end{aligned}$$

where

$$\begin{aligned}
\Xi_{n,k}(u) &= \frac{(1-u)^{k-1}}{2u}\sum_{i=1}^{n-1}\frac{[-k\ln\big(1-u\big)]^i}{i!} - \frac{1}{2u}\sum_{i=1}^{k-1}\binom{k-1}{i}u^i(1-u)^{k-1-i} \\
&= \frac{(1-u)^{k-1}}{2u}\sum_{i=1}^{n-1}\frac{[-k\ln\big(1-u\big)]^i}{i!} - \frac{1-(1-u)^{k-1}}{2u} \\
&= \frac{(1-u)^{k-1}\sum_{i=0}^{n-1}\frac{[-k\ln(1-u)]^i}{i!}-1}{2u}
\end{aligned}$$

$\big($cf (4.1.3)$\big)$. Finally, for arbitrary $\mathbf{c} \in \mathbb{R}^n$, we get

$$\mathbb{E}\left[\sum_{i=1}^{n} c_i(R_{i,k}-\mu)\right] = \int_{-\infty}^{\infty} 2F(x)\big[1-F(x)\big]\Xi_{\mathbf{c},k}\big(F(x)\big)dx, \tag{4.1.10}$$

with

$$\begin{aligned}\Xi_{\mathbf{c},k}(u) &= \sum_{i=1}^{n} c_i \Xi_{i,k}(u) = \frac{(1-u)^{k-1}\sum_{i=1}^{n} c_i \sum_{j=0}^{i-1} \frac{[-k\ln(1-u)]^j}{j!} - \sum_{i=1}^{n} c_i}{2u} \\ &= \frac{(1-u)^{k-1}\sum_{j=0}^{n-1}\left(\sum_{i=j+1}^{n} c_i\right)\frac{[-k\ln(1-u)]^j}{j!} - \sum_{i=1}^{n} c_i}{2u} = \frac{\xi_{\mathbf{c},k}(u)}{2u}\end{aligned}$$

(cf (4.1.5) and (4.1.4)). Inequalities (4.1.6) are immediate consequences of (4.1.9) and (4.1.10).

Now we verify the conditions of getting the equality in the right-hand side inequality of (4.1.6). The arguments justifying the lower bounds attainability are similar. Suppose first that $\Xi_{\mathbf{c},k}(u_1) = \sup_{0<u<1} \Xi_{\mathbf{c},k}(u)$ for some $0 < u_1 < 1$. The equality

$$\int_{-\infty}^{\infty} 2F(x)\big[1-F(x)\big]\Xi_{\mathbf{c},k}\big(F(x)\big)dx = \sup_{0<u<1} \Xi_{\mathbf{c},k}(u) \int_{-\infty}^{\infty} 2F(x)\big[1-F(x)\big]dx \tag{4.1.11}$$

holds iff either $F(x) = 0$ or $F(x) = 1$ or $\Xi_{\mathbf{c},k}(F(x)) = \Xi_{\mathbf{c},k}(u_1)$ for almost all $x \in \mathbb{R}$. The conditions are satisfied by any two-point distribution function

$$F_{u_1}(x) = \begin{cases} 0, & x < a, \\ u_1, & a \le x < b, \\ 1, & x \ge b, \end{cases} \qquad a < b,$$

that assigns probability u_1 to the smaller point a of its support. We have

$$\begin{aligned}\mathbb{E}_m R_{i,k} = \int_{-\infty}^{\infty} x\, F_{i,k}^U(F_m(dx)) &\nearrow \mathbb{E}_{u_1} R_{i,k} = \int_{-\infty}^{\infty} x\, F_{i,k}^U(F_{u_1}(dx)) < \infty, \\ \mathbb{E}_m X_1 = \int_{-\infty}^{\infty} xF_m(dx) &\nearrow \mathbb{E}_{u_1} X_1 = \int_{-\infty}^{\infty} xF_{u_1}(dx) < \infty, \\ \mathbb{E}_m X_{i:2} = \int_{-\infty}^{\infty} xH_{i:2}(F_m(dx)) &\nearrow \mathbb{E}_{u_1} X_{i:2} = \int_{-\infty}^{\infty} xH_{i:2}(F_{u_1}(dx)) < \infty, \quad i = 1, 2,\end{aligned}$$

as $m \to \infty$, where $H_{1:2}(u) = 1-(1-u)^2$ and $H_{2:2}(u) = u^2$, $0 < u < 1$, are the distribution functions of the minimum and maximum of two i.i.d. standard uniform random variables.

Therefore

$$\begin{aligned}\lim_{m\to\infty}\frac{\mathbb{E}_m\sum_{i=1}^n c_i(R_{i,k}-X_1)}{\mathbb{E}_m(X_{2:2}-X_{1:2})} &= \frac{\mathbb{E}_{u_1}\sum_{i=1}^n c_i(R_{i,k}-X_1)}{\mathbb{E}_{u_1}(X_{2:2}-X_{1:2})}\\ &= \frac{1}{\Delta_{u_1}}\int_{-\infty}^{\infty}2F_{u_1}(x)[1-F_{u_1}(x)]\Xi_{\mathbf{c},k}\big(F_{u_1}(x)\big)dx\\ &= \sup_{0<u<1}\Xi_{\mathbf{c},k}(u),\end{aligned}$$

$\big($cf (4.1.9) and (4.1.10)$\big)$, as claimed.

Assume now that $\sup_{0<u<1}\Xi_{\mathbf{c},k}(u)=\lim_{u\searrow 0}\Xi_{\mathbf{c},k}(u)$. Replacing u_1 of the previous paragraph by arbitrary $0<u<1$, and setting $F_{u,m}=uF_{m,a}+(1-u)F_{m,b}$, we obtain

$$\mathbb{E}_{u,m}\sum_{i=1}^n c_i(R_{i,k}-\mu)\to\Xi_{\mathbf{c},k}(u)\int_{-\infty}^{\infty}2F_u(x)\big[1-F_u(x)\big]dx.$$

Replacing fixed u by elements of a sequence $u_m\searrow 0$, we finally get

$$\frac{\mathbb{E}_{u_m,m}\sum_{i=1}^n c_i(R_{i,k}-\mu)}{\Delta_{u_m,m}}\nearrow\sup_{0<u<1}\Xi_{\mathbf{c},k}(u).$$

We proceed in a similar way if $\sup_{0<u<1}\Xi_{\mathbf{c},k}(u)=\lim_{u\nearrow 1}\Xi_{\mathbf{c},k}(u)$. $\square$

Remark 7. *It is natural to assume that $c_n\neq 0$. Then for $k=1$ and $n\geq 2$, function $\Xi_{\mathbf{c},k}$ is unbounded in the left neighborhood of 1. It tends either to $+\infty$ or to $-\infty$ there, and the sign coincides with the sign of $b_n=c_n$. It is clear that $\mathbb{E}|X_1|<\infty$ implies $\Delta=\mathbb{E}(X_{2:2}-X_{1:2})<\infty$. Nagaraja (1978) (see also Arnold et al, 1998, p. 29) constructed parent distribution functions such that $\mathbb{E}|X_1|<\infty$ and $\mathbb{E}R_{n-1,1}<\infty$, but $\mathbb{E}R_{n,1}=+\infty$. This justifies the claim that in the case of the 1st records, there is no finite upper (lower) bound for $\mathbb{E}\sum_{i=1}^n c_i(R_{i,k}-\mu)/\Delta$, when $c_n>0$ ($c_n<0$, respectively). However, it may be surprising that we can get arbitrarily large value (positive or negative one) even if we restrict ourselves to very simple parent distributions with arbitrarily small supports.*

If $k>1$, finiteness of the population mean implies that of all kth records. Note that our bounds are also finite under the assumption.

Remark 8. *There are numerous possibilities of modifying the sequences of distributions attaining the bounds. In the construction $F_{u_1,m}=u_1F_{m,a}+(1-u_1)F_{m,b}$, the sequences of uniform distribution functions $F_{m,a},F_{m,b}$, $m\in\mathbb{N}$, can be substituted with any sequences of continuous distribution functions converging weakly to degenerate ones F_a, F_b concentrated at a and b, respectively. Also, fixed u_1, a and b, can be replaced by sequences u_m,a_m,b_m, with the* *only restrictions that $u_n \to u_1$, and $a_m < b_m$. Moreover, particular $\Xi_{\mathbf{c},k}$ may have multiple extremes. For instance, if $0 < u_1 < \ldots < u_r < 1$ are some arguments maximizing $\Xi_{\mathbf{c},k}$ (not necessarily all), then the equality in (4.1.11) holds for $F = u_1 F_{a_0} + \sum_{i=1}^{r-1}(u_{i+1} - u_i)F_{a_i} + (1-u_r)F_{a_r}$ for some $a_0 < \ldots < a_r$. In consequence, the upper bound is also attained for any sequence of continuous parent distribution functions tending weakly to the above $(r+1)$-point distribution function. Similar modifications can be used if the extremes of (4.1.5) are attained in the limit.*

4.2 Single kth record values

In this section we specify sharp bounds of Theorem 5 for the most practically important cases of single kth record values and the differences of various kth records. By the theorem, the bounds in the first case coincide with extreme values of functions (4.1.3). Note that their derivatives vanish iff

$$\begin{aligned}
\chi_{n,k}(u) &= 2u^2\Xi'_{n,k}(u) = u\xi'_{n,k}(u) - \xi_{n,k}(u) \\
&= u(1-u)^{k-2}\left[\sum_{i=0}^{n-2}\frac{[-k\ln(1-u)]^i}{i!} - (k-1)\frac{[-k\ln(1-u)]^{n-1}}{(n-1)!}\right] \\
&- (1-u)^{k-1}\sum_{i=0}^{n-1}\frac{[-k\ln(1-u)]^i}{i!} + 1 = 0. \qquad (4.2.1)
\end{aligned}$$

We do not treat here the first values of kth records $R_{1,k}$, because they coincide with the first order statistics $X_{1:k}$, and the respective evaluations were presented in the second chapter.

Proposition 12. *Let $X_1, X_2, \ldots$ be i.i.d. with continuous distribution function with finite expectation $\mu = \mathbb{E}X_1$, and Gini mean difference $\Delta = \mathbb{E}|X_1 - X_2|$. We also assume that $\mathbb{E}|R_{n,k}| < \infty$. Then, for various natural $n \geq 2$ and $k \geq 1$, we have the following sharp bounds.*

(i) For $n \geq 2$ and $k = 1$, yields

$$\frac{1}{2} = \Xi_{n,1}(0+) \leq \frac{\mathbb{E}(R_{n,1} - \mu)}{\Delta} \leq \Xi_{n,1}(1-) = \infty.$$

(ii) If $n = k = 2$, then

$$-\frac{1}{2} = \Xi_{2,2}(1-) \leq \frac{\mathbb{E}(R_{2,2} - \mu)}{\Delta} \leq \Xi_{2,2}(0+) = \frac{1}{2}.$$

(iii) For $n \geq 3$ and $k = 2$

$$-\frac{1}{2} = \lim_{u \nearrow 1^-} \Xi_{n,2}(1-) \leq \frac{\mathbb{E}(R_{n,2} - \mu)}{\Delta} \leq \Xi_{n,2}(u_1) > \frac{1}{2},$$

where $u_1 \in (0, 1)$ is the unique solution to the particular version of equation (4.2.1) with $k = 2$.

(iv) For $n = 2$ and $k \geq 3$

$$-\frac{1}{2} > \Xi_{2,k}(u_1) \leq \frac{\mathbb{E}(R_{2,k} - \mu)}{\Delta} \leq \Xi_{2,k}(0+) = \frac{1}{2},$$

where $u_1 \in (0, 1)$ is the unique solution to the particular version of (4.2.1) with $n = 2$.

(v) For $k \geq 3$ and $n \geq 3$, we have

$$-\frac{1}{2} > \Xi_{n,k}(u_2) \leq \frac{\mathbb{E}(R_{n,k} - \mu)}{\Delta} \leq \Xi_{n,k}(u_1) > \frac{1}{2},$$

with $0 < u_1 < u_2 < 1$ being the only two solutions to (4.2.1).

For brevity of presentation, we do not describe precisely attainability conditions. E.g., writing that for some parameters n and k, the upper bound (or lower one) is equal to $\Xi_{n,k}(u_1)$ for some uniquely specified u_1, we refer to Theorem 5, where a sequence of mixtures $F_m = u_1 F_{m,a} + (1 - u_1) F_{m,b}$ of uniform distributions attaining the bound in the limit is described. Similarly, notations $\Xi_{n,k}(0+)$ and $\Xi_{n,k}(1-)$ mean that the extreme values of $\Xi_{n,k}$ are attained in the limit as $u \searrow 0$ and $u \nearrow 1$, respectively, and the conditions of attainability can be found again in Theorem 5. We also refer to Remark 8 for their possible relaxations. We adhere to this convention later on as well.

In the proof of Proposition 12 and some further results we use the following elementary lemma.

Lemma 7. *Let $\psi \colon (a, b) \to \mathbb{R}$, $0 \leq a < b$, be twice differentiable function, $\Psi(x) = \psi(x)/x$ and $\chi(x) = x^2 \Psi'(x) = x\psi'(x) - \psi(x)$ with $\chi'(x) = x\psi''(x)$. We have the following.*

(i) If ψ is positive and decreasing, then Ψ decreases.

(ii) If ψ is negative and increasing, then Ψ increases.

(iii) Assume that ψ is convex.

(a) If $\lim_{x \nearrow b^-} \chi(x) \leq 0$, then Ψ is decreasing.

(b) If $\lim_{x\searrow a^+}\chi(x)\geq 0$*, then* Ψ *is increasing.*

(c) If $\lim_{x\searrow a^+}\chi(x) < 0 < \lim_{x\nearrow b^-}\chi(x)$*, then there exists* $c\in(a,b)$ *such that* Ψ *decreases on* $(a,c]$ *and increases on* $[c,b)$*.*

(iv) Suppose that ψ *is concave.*

(a) If $\lim_{x\searrow a^+}\chi(x)\leq 0$*, then* Ψ *is decreasing.*

(b) If $\lim_{x\nearrow b^-}\chi(x)\geq 0$*, then* Ψ *is increasing.*

(c) If $\lim_{x\searrow a^+}\chi(x) > 0 > \lim_{x\nearrow b^-}\chi(x)$*, then there exists* $c\in(a,b)$ *such that* Ψ *increases on* $(a,c]$ *and decreases on* $[c,b)$*.*

Function $\Psi(x)=\frac{\psi(x)}{x}$ represents the slope of the straight line passing through the origin of the real plane, and the graph of ψ at x. This is increasing (decreasing) there if the slope is less (greater) than that of of the line tangent to ψ at x. Function ψ is called starshaped (antistarshaped) if $\frac{\psi(x)}{x}$ is nondecreasing (nonincreasing, respectively).

Proof. (i),(ii) By definition, $\operatorname{sgn}\big(\Psi'(x)\big)=\operatorname{sgn}\big(\chi(x)\big)$. Assume that ψ is positive and decreasing. Then $\chi(x)=x\psi'(x)-\psi(x)<0$ for all $x\in(a,b)$, $\Psi'(x)<0$, and Ψ decreases. Similarly, if ψ is negative and increasing, then $\chi(x)=\psi'(x)x-\psi(x)>0$, and hence Ψ increases.
(iii) We have $\chi'(x)=x\psi''(x)>0$. Under assumption that ψ is convex, function $\chi(x)$ increases for all $a<x<b$. Accordingly, if $\lim_{x\nearrow b^-}\chi(x)\leq 0$, then $\chi(x)<0$ for all $x\in(a,b)$, and so Ψ is decreasing. If $\lim_{x\searrow a^+}\chi(x)\geq 0$, then $\chi(x)>0$ and Ψ is increasing. Finally, when $\lim_{x\searrow a^+}\chi(x)<0<\lim_{x\nearrow b^-}\chi(x)$, then by Darboux's theorem, there exists $a<c<b$ such that Ψ first decreases on $(a,c]$, and then increases on $[c,b)$.
(iv) The proof is analogous to that of (iii). □

Proof of Proposition 12. By Theorem 5, it suffices to determine the extremes of (4.1.3). We first examine variability of its numerator (4.1.2). We immediately check that $\xi_{n,k}(0+)=0$ for all $k\geq 1$ and $n\geq 2$ (and for $n=1$ as well, but we do not consider the case here). Also,

$$\xi_{n,k}(1-)=\begin{cases}+\infty, & k=1,\\ -1, & k\geq 2,\end{cases}\qquad n\geq 2.$$

We further have

$$\xi'_{n,k}(u)=(1-u)^{k-2}\left[\sum_{i=0}^{n-2}\frac{[-k\ln(1-u)]^i}{i!}-(k-1)\frac{[-k\ln(1-u)]^{n-1}}{(n-1)!}\right]. \tag{4.2.2}$$

If $k=1$, the last term in the square brackets vanishes, and so $\xi'_{n,1}(u)>0$. When $k\geq 2$, due to Lemma 2, (4.2.2) is first positive, and then negative. In consequence, for $k\geq 2$ function

(4.1.2) is first increasing, and ultimately decreasing. Note that the function is necessarily concave about its maximum, because it is smooth.

The second derivative amounts to

$$\begin{aligned}\xi''_{n,k}(u) &= (1-u)^{k-3}\left[\sum_{i=0}^{n-3} 2\frac{[-k\ln(1-u)]^i}{i!} - (k^2-2)\frac{[-k\ln(1-u)]^{n-2}}{(n-2)!}\right.\\ &+ \left.(k-1)(k-2)\frac{[-k\ln(1-u)]^{n-1}}{(n-1)!}\right]. \qquad (4.2.3)\end{aligned}$$

The sum in the brackets does not appear for $n = 2$. The middle term is positive for $k = 1$, and negative otherwise. The last one vanishes for $k = 1$ and 2. Applying Lemma 2, we obtain the following conclusions. Function (4.2.3) is positive for $k = 1$ and $n \geq 2$, and negative for $k = n = 2$. It is first negative, and then positive for $n = 2$ and $k \geq 3$. For $k = 2$ and $n \geq 3$, it is consecutively positive, and negative. And finally for $k, n \geq 3$, the sign order is $+-+$. Notice that negative part cannot be dropped here, because $\xi_{n,k}$ has a concavity region about its global maximum.

Summing up, we arrived to the following conclusions. If $k = 1$ and $n \geq 2$, function (4.1.2) convexly increases from 0 at 0 to $+\infty$ at 1. For $k = n = 2$, (4.1.2) is increasing-decreasing, and concave everywhere. When $k = 2$ and $n \geq 3$, it is first convex increasing, then concave increasing, and finally concave decreasing. For $n = 2$ and $k \geq 3$, it is concave increasing on the left, concave decreasing in the center, and convex decreasing on the right. In all the remaining cases $k, n \geq 3$, the function is consecutively convex increasing, concave increasing, concave decreasing, and convex decreasing.

Now we are in a position to analyze variability of (4.1.3) which is our main task. We start with calculating limit values of (4.1.3) at the end-points 0 and 1. By the de l'Hospital rule, for all $n \geq 2$,

$$\begin{aligned}\Xi_{n;k}(0+) &= \lim_{u\nearrow 0}\frac{(1-u)^{k-1}-1}{2u} + \sum_{i=1}^{n-1}\lim_{u\nearrow 0}\frac{[-k\ln(1-u)]^i}{2ui!}\\ &= \lim_{u\nearrow 0}\frac{(1-k)(1-u)^{k-2}}{2} + \sum_{i=0}^{n-2}\lim_{u\nearrow 0}\frac{k[-k\ln(1-u)]^i}{2(1-u)i!}\\ &= \frac{1-k}{2} + \frac{k}{2} = \frac{1}{2}.\end{aligned}$$

Also,

$$\Xi_{n,k}(1-) = \begin{cases} +\infty, & k = 1, \\ -\frac{1}{2}, & k \geq 2, \end{cases} \qquad n \geq 2,$$

Knowing the shapes of (4.1.2), and using Lemma 7 with $\psi(u) = \xi_{n,k}(u)$, $\Psi(u) = 2\Xi_{n,k}(u)$, $\chi(u) = \chi_{n,k}(u) = 2u^2\Xi'_{n,k}(u) = u\xi'_{n,k}(u) - \xi_{n,k}(u)$, $0 < u < 1$, we are able to describe monotonicity properties of (4.1.3).

Analysis of the case $k = 1 < n$ is the simplest one. We have $\chi_{n,1}(0+) = 0$, and function $\xi_{n,1}$ convexly increases from $\xi_{n,1}(0+) = 0$ to $\xi_{n,1}(1-) = +\infty$. By Lemma 7(iiib), $\Xi_{n,1}$ increases from the $\Xi_{n,1}(0+) = \frac{1}{2}$ to $\Xi_{n,1}(1-) = +\infty$.

We proceed to $k \geq 2$ and consider the most sophisticated case with $k, n \geq 3$. For the other ones, we refer to some arguments presented here. We assume that $\xi_{n,k}$ is convex increasing on $(0, a)$, concave increasing on (a, b), concave decreasing on (b, c), and convex decreasing on $(c, 1)$ for some $0 < a < b < c < 1$, and $\xi_{n,k}(d) = 0$ for some $b < d < 1$. Note that $\chi_{n,k}(0+) = \lim_{u\nearrow 0} u\left[\xi'_{n,k}(u) - \frac{\xi_{n,k}(u)}{u}\right] = 0$. By Lemma 7(iiib), $\Xi_{n,k}$ is increasing on $(0, a)$. We have $\chi_{n,k}(a) > 0$, because the line tangent to $\xi_{n,k}$ at the inflexion point a runs below the line $\frac{\xi_{n,k}(a)}{a}u$ joining the origin point with $(a, \xi_{n,k}(a))$ on $(0, a)$, and above on $(a, 1)$, which means that it has a greater slope. We also have $\chi_{n,k}(b) = -\xi_{n,k}(b) < 0$ at the maximum point b. Owing to Lemma 7(ivc), there is a point $a < u_1 < b$ such that $\Xi_{n,k}$ increases on (a, u_1) and decreases on (u_1, b). By Lemma 7(i), $\Xi_{n,k}$ decreases on (b, d). Suppose now that $d < c$. Then $\chi_{n,k}(d) = d\xi'_{n,k}(d) < 0$, and so $\Xi_{n,k}$ still decreases on (d, c) by Lemma 7(iva). Comparing the slopes of straight lines $\xi_{n,k}(c) + \xi'_{n,k}(c)(u - c)$ and $\frac{\xi_{n,k}(c)}{c}u$, we conclude that $\chi_{n,k}(c) < 0$. We also observe that $\chi_{n,k}(1-) = -\xi_{n,k}(1-) = 1 > 0$. With use of the last claim of Lemma 7(iii), we conclude that $\Xi_{n,k}$ decreases on (c, u_2) and increases on $(u_2, 1)$ for some $c < u_2 < 1$. If $d \geq c$, we again recall the relations $\chi_{n,k}(d) < 0 < \chi_{n,k}(1-)$ and Lemma 7(iiic) for deducing that there exists $d < u_2 < 1$ such that $\Xi_{n,k}$ decreases on (d, u_2) and increases on $(u_2, 1)$. Combining the above results, we arrive to the following conclusion: $\Xi_{n,k}$ first increases from $\frac{1}{2}$ at $0+$ to $\Xi_{n,k}(u_1) > \frac{1}{2}$, and the decreases to $\Xi_{n,k}(u_2) < -\frac{1}{2}$, and finally increases to $-\frac{1}{2}$ at $1-$. This implies that the global maximum and minimum are attained at u_1 and u_2, respectively, which are the only local extremes of $\Xi_{n,k}$ in $(0, 1)$.

If $k = 2 < n$, function $\xi_{n,2}$ does not have a decreasing convex part at the left neighborhood of 1. We can just put $c = 1 > d > b$, and repeat the above reasoning omitting analysis of the functions on the interval $(c, 1)$, when $d < c$. Case $c \leq d < 1$ is impossible then. In consequence, we observe that $\Xi_{n,2}$ increases from $\Xi_{n,2}(0+) = \frac{1}{2}$ to $\Xi_{n,2}(u_1) > \frac{1}{2}$, and decreases to $\Xi_{n,2}(1-) = -\frac{1}{2}$. The global extremes are $\Xi_{n,2}(u_1) > \frac{1}{2}$, and $\Xi_{n,2}(1-) = -\frac{1}{2}$.

For $n = 2 < k$, $\xi_{2,k}$ is deprived of the increasing convex part on the left. However, then we still have $\chi_{2,k}(0+) = 0$, and we can use the argument of Lemma 7(iva) to conclude that $\Xi_{2,k}$ is decreasing on (a, b) with $a = 0$. Then we repeat the reasoning of the previous paragraph applied to studying functions $\xi_{n,k}$, $\chi_{n,k}$, and $\Xi_{n,k}$ on the interval $(b, 1)$. Accordingly, we conclude that $\Xi_{2,k}$ decreases from $\Xi_{2,k}(0+) = \frac{1}{2}$ to $\Xi_{2,k}(u_1) < -\frac{1}{2}$, and increases to $\Xi_{2,k}(1-) = -\frac{1}{2}$. This means that $-\frac{1}{2} > \Xi_{2,k}(u_1) \leq \Xi_{2,k}(u) < \Xi_{2,k}(0+) = \frac{1}{2}$.

For $k = n = 2$, we can reduce the arguments as in the two above cases by removing from analysis two convexity intervals of $\xi_{n,k}$ appearing in both the ends of the unit interval. As a result, we observe that $\Xi_{2,2}$ decreases from $\Xi_{2,2}(0+) = \frac{1}{2}$, to $\Xi_{2,2}(1-) = -\frac{1}{2}$ which are clearly the extreme values of the function. This completes the proof of Proposition 12. $\square$

Table 4.1: Upper bounds on expectations nth values of 2nd records $\frac{\mathbb{E}R_{n,2}-\mu}{\Delta}$, and upper and lower bounds on expectations of 8th records $\frac{\mathbb{E}R_{n,8}-\mu}{\Delta}$ for $n = 3, \dots, 11$.

n	$u_1(n,2)$	$\Xi_{n,2}(u_1)$	$u_1(n,8)$	$\Xi_{n,8}(u_1)$	$u_2(n,8)$	$\Xi_{n,8}(u_2)$
3	0.53864	0.67515	0.00612	0.50151	0.49172	-0.82907
4	0.85953	1.27417	0.05275	0.51740	0.63022	-0.69988
5	0.95425	2.48879	0.12728	0.54943	0.72995	-0.63051
6	0.98408	4.81797	0.21163	0.59439	0.80242	-0.58866
7	0.99425	9.23834	0.29654	0.65089	0.85534	-0.56167
8	0.99788	17.6289	0.37741	0.71872	0.89407	-0.54356
9	0.99921	33.6037	0.45206	0.79835	0.92244	-0.53108
10	0.99971	64.1276	0.51968	0.89066	0.94324	-0.52232
11	0.99989	122.652	0.58017	0.99688	0.95848	-0.51611

Table 4.1 presents numerical values of upper bounds $\Xi_{n,k}(u_1)$ on expectations of kth records for $k = 2, 8$ and $n = 3, \dots, 11$, and the values of lower bounds $\Xi_{n,8}(u_2)$ on expectations of 8th records for $n = 3, \dots, 11$. They are accompanied by respective arguments $u_1 = u_1(n,k)$ for which $\Xi_{n,k}$ attain their maxima, and $u_2 = u_2(n,k)$ for which $\Xi_{n,k}$ attain the minima. The lower bounds on the expectations of second records amount to $\Xi_{n,2}(1-) = -\frac{1}{2}$. The arguments of the extremes allow us to recover the distributions attaining the bounds. It is obvious that $\Xi_{n,k}(u_1)$, and $\Xi_{n,8}(u_2)$ increase as n increases from 3 to 11 for both $k = 2$ and 8. It is worth noting that $u_1(n,k)$, $u_2(n,k)$ do so as well.

4.3 Differences of kth record values

Now we evaluate the expectations of differences of kth record values $\mathbb{E}(R_{n,k} - R_{m,k})$, $1 \leq m < n$. By Theorem 5, the problem boils down to finding the extremes of functions

$$\begin{aligned} \Xi_{m,n;k}(u) &= \Xi_{n,k}(u) - \Xi_{m,k}(u) = \frac{\xi_{n,k}(u) - \xi_{m,k}(u)}{2u} \\ &= \frac{(1-u)^{k-1}}{2u} \sum_{i=m}^{n-1} \frac{\big[-k\ln(1-u)\big]^i}{i!}, \qquad 0 < u < 1. \end{aligned} \tag{4.3.1}$$

The local extremes of the functions (if they exist) satisfy the equalities

$$\begin{aligned} \frac{\chi_{m,n;k}(u)}{(1-u)^{k-2}} &= \frac{\chi_{n,k}(u) - \chi_{m,k}(u)}{(1-u)^{k-2}} = \frac{2u^2\Xi'_{m,n;k}(u)}{(1-u)^{k-2}} \\ &= \frac{u[\xi'_{n,k}(u) - \xi'_{m,k}(u)] - [\xi_{n,k}(u) - \xi_{m,k}(u)]}{(1-u)^{k-2}} \\ &= ku\frac{[-k\ln(1-u)]^{m-1}}{(m-1)!} + 2u\sum_{i=m}^{n-2} \frac{[-k\ln(1-u)]^i}{i!} \\ &- (k-2)u\frac{[-k\ln(1-u)]^{n-1}}{(n-1)!} - \sum_{i=m}^{n-1} \frac{[-k\ln(1-u)]^i}{i!} = 0. \end{aligned} \tag{4.3.2}$$

Proposition 13. *Under the assumptions of Proposition 12, the following statements hold true.*

(i) If $k = m = 1$ and $n \geq 2$,

$$\frac{1}{2} = \Xi_{1,n;1}(0+) \leq \frac{\mathbb{E}(R_{n,1} - R_{1,1})}{\Delta} \leq \Xi_{1,n;1}(1-) = +\infty.$$

(ii) If $k = 1$ and $2 \leq m < n$, then

$$0 = \Xi_{m,n;1}(0+) \leq \frac{\mathbb{E}(R_{n,1} - R_{m,1})}{\Delta} \leq \Xi_{m,n;1}(1-) = +\infty.$$

(iii) If either $k = n = 2$ and $m = 1$, or $k \geq 3$, $n = 2, 3$, and $m = 1$, then

$$0 = \Xi_{1,n;k}(1-) \leq \frac{\mathbb{E}(R_{n,k} - R_{1,k})}{\Delta} \leq \Xi_{1,n;k}(0+) = \frac{k}{2}.$$

(iv) If $k = 2, 3$, $m = 1$ and $n \geq k + 1$, then

$$0 = \Xi_{1,n;k}(1-) \leq \frac{\mathbb{E}(R_{n,k} - R_{1,k})}{\Delta} \leq \Xi_{1,n;k}(u_1) > \frac{k}{2},$$

where $u_1 \in (0, 1)$ is only one solution of equation (4.3.2).

(v) For $k = 2, 3$ with $2 \leq m < n$, and for $k \geq 4$, with $m \geq 2$ and $n = m+1, m+2$, we have

$$0 = \Xi_{m,n;k}(0+) = \Xi_{m,n;k}(1-) \leq \frac{\mathbb{E}(R_{n,k} - R_{m,k})}{\Delta} \leq \Xi_{m,n;k}(u_1) > 0,$$

where $u_1 \in (0, 1)$ is the unique solution to (4.3.2).

(vi) For $k \geq 4$, $m = 1$ and $n \geq 4$, equation (4.3.2) has either no solutions in $(0, 1)$, and then

$$0 = \Xi_{1,n;k}(1-) \leq \frac{\mathbb{E}(R_{n,k} - R_{1,k})}{\Delta} \leq \Xi_{1,n;k}(0+) = \frac{k}{2},$$

or it has two solutions $0 < u_1 < u_2 < 1$, and then

$$0 = \Xi_{1,n;k}(1-) \leq \frac{\mathbb{E}(R_{n,k} - R_{1,k})}{\Delta} \leq \max\left\{\frac{k}{2}, \Xi_{1,n;k}(u_2)\right\} = \max\left\{\Xi_{1,n;k}(0+), \Xi_{1,n;k}(u_2)\right\}.$$

(vii) For all $k \geq 4$, $m \geq 2$ and $n \geq m + 3$, either (4.3.2) has a unique solution u_1 in $(0, 1)$, and then

$$0 = \Xi_{m,n;k}(0+) = \Xi_{m,n;k}(1-) \leq \frac{\mathbb{E}(R_{n,k} - R_{m,k})}{\Delta} \leq \Xi_{m,n;k}(u_1) > 0,$$

or it has three solutions $u_1 < u_2 < u_3$ there, and, in consequence,

$$0 = \Xi_{m,n;k}(0+) = \Xi_{m,n;k}(1-) \leq \frac{\mathbb{E}(R_{n,k} - R_{m,k})}{\Delta} \leq \max\{\Xi_{m,n;k}(u_1), \Xi_{m,n;k}(u_3)\}.$$

The bounds for the most interesting subcase of kth record spacings $R_{m+1,k} - R_{m,k}$ for particular pairs of parameters $k = m = 1$, $k = 1 < m$, $m = 1 < k$ and $k, m \geq 2$ can be immediately concluded from points (i), (ii), (iii), and (v) of Proposition 13, respectively.

Proof. The idea is similar to the previous proof. We first analyze the numerator

$$\xi_{m,n;k}(u) = (1-u)^{k-1} \sum_{i=m}^{n-1} \frac{[-k \ln(1-u)]^i}{i!}. \tag{4.3.3}$$

of (4.3.1). We immediately check that $\xi_{m,n;k}(0+) = 0$ for all possible m, n, and k, and $\xi_{m,n;k}(1-) = +\infty$ when $k = 1$, and 0 otherwise. Furthermore

$$\begin{aligned} \xi'_{m,n;k}(u) &= (1-u)^{k-2} \left[k \frac{[-k\ln(1-u)]^{m-1}}{(m-1)!} \right. \\ &+ \left. \sum_{i=m}^{n-2} \frac{[-k\ln(1-u)]^i}{i!} - (k-1) \frac{[-k\ln(1-u)]^{n-1}}{(n-1)!} \right] \end{aligned}$$

If $k = 1$, the last term vanishes, and (4.3.3) is increasing on the unit interval. By VDP of Lemma 2, the function is first increasing and then decreasing for all $k \geq 2$.

Analysis of the second derivative

$$\xi''_{m,n;k}(u) = (1-u)^{k-3} \left[k^2 \frac{[-k\ln(1-u)]^{m-2}}{(m-2)!} - k(k-3) \frac{[-k\ln(1-u)]^{m-1}}{(m-1)!} \right.$$
$$+ \left. \sum_{i=m}^{n-3} 2 \frac{[-k\ln(1-u)]^i}{i!} - (k^2-2) \frac{[-k\ln(1-u)]^{n-2}}{(n-2)!} + (k-1)(k-2) \frac{[-k\ln(1-u)]^{n-1}}{(n-1)!} \right] \quad (4.3.4)$$

is more complex. The coefficient of the first term vanishes for $m = 1$, and is positive for $m \geq 2$. That of the second one is positive for $k = 1, 2$, equal to 0 for $k = 3$ and negative for other $k \geq 4$. If $n = m+1, m+2$, the sum is dropped, and its summands are positive otherwise. The penultimate ingredient has a positive coefficient for $k = 1$, and negative one for $k \geq 2$. And that of the last one is either 0 when $k = 1, 2$ or positive otherwise.

Applying Lemma 2, and taking into account the fact that a smooth function has to be concave about its local maximum, we arrive to the following conclusions. If $k = 1$, then (4.3.4) is positive. Therefore (4.3.3) convexly increases from $\xi_{m,n;k}(0+) = 0$ to $\xi_{m,n;k}(1-) = +\infty$. Otherwise the function is increasing-decreasing, and vanishes at 0 and 1.

If $k = 2$ and $m = 1$, $n = 2$, it is concave in $(0, 1)$. If $k = 2$ and either $m = 1$ with $n \geq 3$ or $n > m \geq 2$, (4.3.4) changes the sign from $+$ to $-$, which means that (4.3.3) is first convex increasing, then concave increasing and finally concave decreasing.

Suppose now that $k = 3$. If $m = 1$ and $n = 2, 3$, then (4.3.4) is negative-positive, and so (4.3.3) is concave increasing, concave decreasing and convex decreasing. Otherwise, i.e., for $m = 1$ with $n \geq 4$, and $n > m \geq 2$, the sign sequence of (4.3.4) is $+-+$. This implies that (4.3.3) is consecutively convex increasing, concave increasing, concave decreasing and convex decreasing at the right end.

Assume finally that $k \geq 4$. Then for $m = 1$ and $n = 2, 3$, the second derivative (4.3.4) is negative-positive, and therefore the original function (4.3.3) is concave increasing, concave decreasing and convex decreasing. If $m = 1$ and $n \geq 4$, the sign order of the combination coefficients in (4.3.4) is $-+-+$. For function (4.3.4) itself, it may reduce to $-+$. Note that in the first case the maximum point of (4.3.3) can belong to either of two its concavity regions. Consequently, we have three possible behaviors of (4.3.3). Firstly, it may be concave increasing, concave decreasing and convex decreasing. Secondly, it may be concave increasing, and, on the region of decrease, it may be consecutively concave, convex, and again concave and convex. The last option is that (4.3.3) is consecutively concave, convex and concave on the interval of increase, and concave and convex in the decrease area. If $m \geq 2$ and $n = m+1, m+2$, function (4.3.4) is first positive, then negative and eventually positive. It follows that in this case (4.3.3) is convex increasing, concave increasing and decreasing, and finally convex decreasing. Lastly, for $m \geq 2$ and $n \geq m+3$, the signs of the combination coefficients are ordered as $+-+-+$. Analysis similar to that of the case $k, n \geq 4$ with $m = 1$ leads to analogous conclusions. We have again three possibilities. The functions are similar, and the only difference is that in each case one should add an interval of convex increase at the beginning.

Now we proceed to analyzing (4.3.1). We have

$$\Xi_{m,n;k}(0+) = \begin{cases} \frac{k}{2}, & m = 1, \\ 0, & m \geq 2, \end{cases}$$

and

$$\Xi_{m,n;k}(1-) = \begin{cases} +\infty, & k = 1, \\ 0, & k \geq 2. \end{cases}$$

Also, $\chi_{m,n;k}(0+) = 0$ for all k, m, and n. This, together with Lemma 7(iiib), imply that for $k = 1$, $\Xi_{m,n;1}$ strictly increases from $\frac{1}{2}$, when $m = 1$ and from 0, when $m \geq 2$ at 0+ to $+\infty$ for all $m \geq 1$ at $1-$, which gives statements (i) and (ii) of the Proposition.

The remaining cases with $k \geq 2$ can be treated in much the same way. Functions $\xi_{m,n;k}$ are first increasing and then decreasing, and tend to 0 as the argument tends to 0 and 1. Respective functions $\chi_{m,n;k}$ are negative at the maximum points of $\xi_{m,n;k}$. Functions $\Xi_{m,n;k}$ are also positive on $(0, 1)$, and vanish at the right end point. Accordingly, 0 provides the sharp lower bound for the differences of all kth records with $k \geq 2$, and they are attained as parameter u converges to 1. Note that this trivial bound is attained for $m \geq 2$ if $u \searrow 0$ as well.

We start with analysis of the most complex case with $k \geq 4$, $m \geq 2$ and $n \geq m+3$. The first option is that (4.3.3) is convex concave and convex which implies that the maximum point belongs to the concavity region. We examine it together with another case that (4.3.3) has two concavity regions, and the maximum is located in the first one. Let $(0, a)$, (a, b) and $(b, 1)$ denote the intervals of convex increase, concave increase, and decrease of the function, respectively. We have $\chi_{m,n;k}(0+) = 0 < \chi_{m,n;k}(a)$, and $\chi_{m,n;k}(b) < 0$. By Lemma 7(iiib) and (ivc), (4.3.1) is increasing on $(0, a)$, and increasing-decreasing on (a, b) with a maximum point at $a < u_1 < b$. By Lemma 7(i), it is also decreasing on $(b, 1)$. Therefore the extreme values of the function are $\Xi_{m,n;k}(0+) = \Xi_{m,n;k}(1-) = 0$ and $\Xi_{m,n;k}(u_1) > 0$.

Note that the analogous arguments are used for $\Xi_{m,n;k}$ with parameters m, n, k such that (4.3.3) is first convex increasing, than concave increasing, and ultimately decreasing, i.e. for $k = 2$ with either $m = 1$ and $n \geq 3$ or $m \geq 2$, for $k = 3$ with either $m = 1$, $n \geq 4$ or $n > m \geq 2$ and for $k \geq 4$ with $m \geq 2$ and $n = m+1, m+2$, which cover cases (iv) and (v) of the Proposition. The only difference between them is that for $m = 1$ function (4.3.1) starts from $\frac{k}{2}$, and then the extreme values are $\Xi_{m,n;k}(1-) = 0$ and $\Xi_{m,n;k}(u_1) > \frac{k}{2}$ (see Proposition 13(iv)), and otherwise $\Xi_{m,n;k}(0+) = 0$ is another possibility for the infimum, and then the maximal value $\Xi_{m,n;k}(u_1) > 0$ does not need to exceed $\frac{k}{2}$ (see Proposition 13(v)).

Let us come back to $k \geq 4$, $m \geq 2$ and $n \geq m+3$, and consider the last case that there are two intervals of convex increase $(0, a)$, and (b, c), say, and two intervals of concave increase (a, b), and (c, d). We certainly have $\chi_{m,n;k}(0+) = 0$, $\chi_{m,n;k}(a) > 0$, and $\chi_{m,n;k}(d) < 0$. Lemma 7(iiib) implies that $\Xi_{m,n;k}$ increases on $(0, a)$. Suppose first that $\chi_{m,n;k}(b) \geq 0$. By Lemma 7(ivb) and (iiib), (4.3.1) is increasing on both (a, b) and (b, c). Convexity of $\xi_{m,n;k}$ on (b, c) implies that $\chi'_{m,n;k}(u) = u\xi''_{m,n;k}(u) > 0$ for $b < u < c$, and so $\chi_{m,n;k}(c) > 0$ as well. Lemma 7(ivb) assures that there is $c < u_1 < d$ such that (4.3.1) is increasing on (c, u_1) and decreasing on (u_1, d). Final decrease of (4.3.1) on $(d, 1)$ is implied by Lemma 7(i). This means that assumption $\chi_{m,n;k}(b) \geq 0$ leads us to the first statement of Proposition 13(vii).

Suppose now that $\chi_{m,n;k}(b) < 0$. Then $\Xi_{m,n;k}$ in increasing on $(0, u_1)$ and decreasing on (u_1, b) for some $a < u_1 < b$ by Lemma 7(iiib) and (ivc). By convexity of $\xi_{m,n;k}$, $\chi_{m,n;k}$ is increasing on (b, c). It may happen that either $\chi_{m,n;k}(c) \leq 0$ or $\chi_{m,n;k}(c) > 0$. Suppose that the first case holds. Then (4.3.1) decreases on (b, c), (c, d), and $(d, 1)$ by Lemma 7(iiia), (iva) and (i), respectively. Again, we conclude that $\Xi_{m,n;k}$ has one local maximum in $(0, 1)$, and the first claim of Proposition 13(vii) holds. The last possibility is that condition $\chi_{m,n;k}(b) < 0$ is accompanied by $\chi_{m,n;k}(c) > 0$. Then except for the local maximum at $a < u_1 < b$, we have a local minimum at $b < u_2 < c$ by Lemma 7(iiic), and another local maximum at $c < u_3 < d$ by Lemma 7(ivc). This is obviously decreasing on $(d, 1)$ by Lemma 7(i). Note that $\Xi_{m,n;k}(u_2) > 0$, because $\Xi_{m,n;k}$ is continuous and positive on $(0, 1)$. Accordingly, the latter statement of Proposition 13(vii) holds.

Analysis of the penultimate case with $k \geq 4$, $m = 1$ and $n \geq 4$ is similar, and we merely outline the main steps of the proof. The only differences are that there is no interval on convex increase in the right neighborhood of 0, and $\Xi_{1,n;k}(0+) = \frac{k}{2}$. We can treat together the cases that (4.3.1) is concave on the whole interval of its increase, whereas the decrease region contains either one or two intervals of convexity. Then $\Xi_{1,n;k}$ is decreasing on both intervals where $\xi_{1,n;k}$ increases and decreases by Lemma 7(iva) and (i), and the first claim of Proposition 13(vi) is valid. Note that in the same way we can treat the cases of Proposition 13(iii) and get the respective conclusion.

Suppose now that the interval of increase $(0, d)$, say, contains one region of convexity (b, c), and two regions of concavity $(0, b)$ and (c, d) (we do not use letter a for the sake of consistency with the previous notation). We have $\chi_{1,n;k}(0+) = 0$, $\chi_{m,n;k}(b) < 0$, and $\chi_{m,n;k}(d) < 0$. If $\chi_{m,n;k}(c) \le 0$, $\Xi_{1,n;k}$ is decreasing on the whole unit interval by Lemma 7(iva), (iiia), again (iva) and (i). If $\chi_{1,n;k}(c) > 0$, then (4.3.1) first decreases, then has a unique local minimum at $b < u_1 < c$, and a unique local maximum at $c < u_2 < d$, and finally decreases by Lemma 7 (ivc), (iiic), (ivc) and (i). Again, we have $\Xi_{1,n;k}(u_1) > \Xi_{1,n;k}(1-) = 0$, but we cannot settle either of two maxima $\Xi_{1,n;k}(0+) = \frac{k}{2}$ and $\Xi_{1,n;k}(u_2)$ is greater. This finishes the proof of case (vi), and of the whole Proposition. □

Table 4.2: Upper bounds on expectations of kth record differences $\mathbb{E}\frac{R_{n,k}-R_{1,k}}{\Delta}$ for $k = 2, 3, 4$ and $n = 4, \ldots, 11$.

n	$u_1(n,2)$	$\Xi_{1,n;2}(u_1)$	$u_1(n,3)$	$\Xi_{1,n,3}(u_1)$	$u_1(n,4)$	$\Xi_{1,n;4}(u_1)$
4	0.85953	1.77417	0.24174	1.51047	0	2
5	0.95425	2.98879	0.59908	1.64906	0	2
6	0.98408	5.31797	0.78565	1.99633	0	2
7	0.99425	9.73834	0.88026	2.58914	0	2
8	0.99788	18.1289	0.93089	3.50031	0.77158	2.13923
9	0.99921	34.1037	0.95925	4.85335	0.84482	2.54721
10	0.99970	64.6276	0.97565	6.83793	0.89233	3.11471
11	0.99989	123.152	0.98532	9.73634	0.92438	3.87706

Table 4.2 contains numerical values of upper bounds $\Xi_{1,n;k}(u_1)$ on the expectations of the differences between the nth and first values of kth records together with respective arguments $u_1 = u_1(n, k)$ for which $\Xi_{1,n;k}$ attains its maximum. We examine $k = 2, 3, 4$ and $n = 4, \ldots, 11$. For calculating the bounds in cases $k = 2, 3$, we applied Proposition 13(iv). For $k = 4$, Proposition 13(vi) was used. Then the first subcase of no local extremes appeared for $n = 4, \ldots, 7$, and the single local maxima of $\Xi_{1,n;4}$ were used for $n = 8, \ldots, 11$. As one can expect, the bounds decrease in rows, and increase in columns. The same tendency concerns the arguments providing the maxima. However, it is quite surprising that as n increases, the points attaining the fast increasing maxima approach very close point 1, where the global infima, equal to 0, are attained.

Table 4.3 presents numerical values of upper bounds $\Xi_{2,n;k}(u_i)$ on the expectations of differences of nth and second values of kth records $\mathbb{E}\frac{R_{n,k}-R_{2,k}}{\Delta}$ with the arguments $u_i = u_i(n, k)$, $i = 1$ or 3, providing the maxima of respective functions $\Xi_{2,n;k}$. We consider parameters

Table 4.3: Upper bounds on on expectations of kth record differences $\mathbb{E}\frac{R_{n,k}-R_{2,k}}{\Delta}$ for $k=2,3$ with $n=3,\dots,11$, and for $k=10$ with $n=13,\dots,21$.

n	$u_1(n,2)$	$\Xi_{2,n;2}(u_1)$	$u_1(n,3)$	$\Xi_{2,n,3}(u_1)$	n	$u_i(n,10)$	$\Xi_{2,n;10}(u_i)$
3	0.79681	0.64761	0.47471	0.54207	13	0.26010	2.08544
4	0.90626	1.49438	0.60992	0.96579	14	0.26018	2.08548
5	0.96101	2.84937	0.72476	1.38507	15	0.26021	2.08549
6	0.98491	5.25240	0.81886	1.87945	16	0.26021	2.08549
7	0.99434	9.70871	0.88772	2.54006	17	0.69271	2.13547
8	0.99789	18.1159	0.93242	3.48013	18	0.72874	2.24568
9	0.99921	34.0981	0.95956	4.84509	19	0.75943	2.37671
10	0.99971	64.6252	0.97571	6.83455	20	0.78605	2.52859
11	0.99989	123.151	0.98533	9.73495	21	0.80936	2.70201

$k=2,3$ with $n=3,\dots,11$ and $k=10$ with $n=13,\dots,21$. Conclusions of Proposition 13 (v) and (vii) were used for $k=2,3$ and $k=10$ respectively. In the latter case, for $n=13,14$, function (4.3.1) has a unique maximum in $(0,1)$, and we use the first statement of Proposition 13(vii). Otherwise it has two local maxima and a minimum between them. However, for $n=15,16$ the global maximum is attained in the first zero of (4.3.2), and for the remaining $n=17,\dots,21$, the last zero provides the global maximum. This explains a significant jump from 0.26021 to 0.69271 in the penultimate column of the Table. Behaviour of the bounds and parameters describing their attainability conditions is like for Table 4.2.

Table 4.4 presents upper bounds $\Xi_{m,m+1;k}(u_1)$ on expectations of kth record spacings $R_{m+1,k}-R_{m,k}$ for $k=2,3,4$ and $m=2,\dots,11$, and respective arguments $u_1=u_1(m,k)$ for which $\Xi_{m,m+1;k}$ attains its maximum. They were established by means of Proposition 13(v). We observe that except for $k=2$, the bounds first decrease and then increase as m increases. The lower bounds for the differences of records presented in Tables 4.2–4.4 amount to 0.

Table 4.4: Upper bounds on expectations of kth record spacings $\mathbb{E}\frac{R_{m+1,k}-R_{m,k}}{\Delta}$ for $k = 2, 3, 4$ and $m = 2, \ldots, 11$.

m	$u_1(m,2)$	$\Xi_{m,m+1;2}(u_1)$	$u_1(m,3)$	$\Xi_{m,m+1;3}(u_1)$	$u_1(m,4)$	$\Xi_{m,m+1;4}(u_1)$
2	0.79681	0.64761	0.47471	0.54208	0.32620	0.58475
3	0.94048	0.94762	0.71317	0.50558	0.54156	0.45015
4	0.98017	1.59328	0.83871	0.58006	0.68538	0.43335
5	0.99302	2.82685	0.90731	0.72982	0.78244	0.46358
6	0.99748	5.15281	0.94588	0.96486	0.84858	0.52656
7	0.99908	9.54491	0.96804	1.31373	0.89404	0.62105
8	0.99966	17.8731	0.98097	1.82275	0.92550	0.75117
9	0.99988	33.7336	0.98860	2.56139	0.94743	0.92468
10	0.99995	64.0592	0.99315	3.63192	0.96278	1.15284
11	0.99998	122.245	0.99587	5.18426	0.97358	1.45096

Chapter 5

Bounds on the variances of linear combinations of kth records

Let $X_1, X_2, \ldots$ be i.i.d. random variables with common continuous distribution function F. For the sequence, we define respective kth record values $R_{1,k}, R_{2,k}, \ldots$. Let $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$ be an arbitrary nonzero vector. Our purpose is to provide bounds for the ratios of variances $\mathbb{V}ar(\sum_{i=1}^{n} c_i R_{i,k})/\mathbb{V}ar\, X_1$ for all possible continuous baseline distribution functions F for which the above variances are finite. Note that finiteness of $\mathbb{V}ar\, X_1$ implies the same for $\mathbb{V}ar\, R_{n,k}$, $n \in \mathbb{N}$, when $k \geq 2$. For the classic records with $k = 1$, condition $\mathbb{V}ar\, X_1 < \infty$ does to suffice for $\mathbb{V}ar\, R_{n,1} < \infty$, $n = 2, 3, \ldots$. Throughout the chapter, writing $\mathbb{V}ar\, Y$ we tacitly assume that this is finite. Below we present upper bounds on variances of arbitrary linear combinations of kth records, and describe conditions of their sharpness. We also determine conditions which imply that respective lower bounds vanish. We first mention sharp lower and upper bounds for single kth record values determined by Klimczak and Rychlik (2004). Then we thoroughly study the case of kth record spacings $R_{m+1,k} - R_{m,k}$.

The literature concerning evaluations of variances of records is very scanty. The first paper devoted to evaluation of variances of records was due to Klimczak and Rychlik (2004) who determined tight lower and upper bounds on variances of single kth record values $\mathbb{V}ar\, R_{n,k}$ measured in the population variance units $\mathbb{V}ar\, X_1$. These results were specified by Jasiński (2016) under some restrictions on parameters n and k.

5.1 Linear combinations of kth record values

For given positive integers n and k, and for a fixed vector $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$, $\sum_{i=1}^n |c_i| > 0$, we define function

$$\begin{aligned}
\Phi_{\mathbf{c},k}(u,v) &= \frac{(1-v)^{k-1}}{u} \left\{ \left[\sum_{j=1}^{n} c_j - (1-u)^k \sum_{i=0}^{n-1} \left(\sum_{j=i+1}^{n} c_j \right) \frac{[-k\ln(1-u)]^i}{i!} \right] \right. \\
&\times \sum_{i=0}^{n-1} \left(\sum_{j=i+1}^{n} c_j \right) \frac{[-k\ln(1-v)]^i}{i!} \\
&- \left. \sum_{1\le i<j\le n} c_i c_j \sum_{p=0}^{j-i-1} \sum_{q=0}^{p} \frac{(-1)^q[-k\ln(1-u)]^{i+q}[-k\ln(1-v)]^{p-q}}{(i-1)!q!(p-q)!(q+i)} \right\} \qquad (5.1.1)
\end{aligned}$$

acting on the triangle $0 < u \le v < 1$. For brevity, parameter n is suppressed in the notation. The diagonal version $\Psi_{\mathbf{c},k}(u)$ of $\Phi_{\mathbf{c},k}(u,u)$ for $0 < v = u < 1$ has much simpler form

$$\begin{aligned}
\Psi_{\mathbf{c},k}(u) &= \frac{(1-u)^{k-1}}{u} \left\{ \sum_{i=0}^{n-1} \left(\sum_{j=i+1}^{n} c_j \right)^2 \frac{[-k\ln(1-u)]^i}{i!} \right. \\
&- \left. (1-u)^k \left[\sum_{i=0}^{n-1} \left(\sum_{j=i+1}^{n} c_j \right) \frac{[-k\ln(1-u)]^i}{i!} \right]^2 \right\}, \qquad (5.1.2)
\end{aligned}$$

because due to the identity

$$\begin{aligned}
&\frac{(p-1)!(q-1)!}{(p+q-1)!} = \int_0^1 u^{p-1}(1-u)^{q-1}du \\
= &\sum_{r=0}^{q-1} \binom{q-1}{r} (-1)^r \int_0^1 u^{p+r-1}du = \sum_{r=0}^{q-1} \frac{(q-1)!(-1)^r}{r!(q-1-r)!(p+r)},
\end{aligned}$$

the last line of (5.1.1) for $u = v$ can be rewritten as

$$\begin{aligned}
&\sum_{j=2}^{n} c_j \sum_{i=1}^{j-1} c_i \sum_{p=0}^{j-i-1} \frac{[-k\ln(1-u)]^{i+p}}{(i-1)!} \sum_{q=0}^{p} \frac{(-1)^q}{q!(p-q)!(q+i)} \\
=& \sum_{j=2}^{n} c_j \sum_{i=1}^{j-1} c_i \sum_{p=0}^{j-i-1} \frac{[-k\ln(1-u)]^{i+p}}{(i+p)!} \\
=& \sum_{j=2}^{n} c_j \sum_{i=1}^{j-1} c_i \sum_{p=i}^{j-1} \frac{[-k\ln(1-u)]^{p}}{p!} \\
=& \sum_{j=2}^{n} c_j \sum_{p=1}^{j-1} \left(\sum_{i=1}^{p} c_i\right) \frac{[-k\ln(1-u)]^{p}}{p!} \\
=& \sum_{p=1}^{n-1} \left(\sum_{j=p+1}^{n} c_j\right)\left(\sum_{i=1}^{p} c_i\right) \frac{[-k\ln(1-u)]^{p}}{p!}.
\end{aligned}$$

Theorem 6. *Suppose that $X_1, X_2, \dots$ is a sequence of i.i.d. random variables with a common continuous distribution function F, say, such that $\mathbb{E}X_1^2$ and $\mathbb{E}R_{n,k}^2$ are finite for fixed $n, k \in \mathbb{N}$. Then for any non-zero $\mathbf{c} \in \mathbb{R}^n$, we have*

$$\frac{\mathbb{V}ar(\sum_{i=1}^{n} c_i R_{i,k})}{\mathbb{V}ar\, X_1} \le \sup_{0<u\le v<1} \Phi_{\mathbf{c},k}(u,v). \tag{5.1.3}$$

Moreover, if

$$\sup_{0<u\le v<1} \Phi_{\mathbf{c},k}(u,v) = \sup_{0<u<1} \Psi_{\mathbf{c},k}(u), \tag{5.1.4}$$

then bound (5.1.3) is sharp. Precisely, we have the following.
(i) If $\sup_{0<u<1} \Psi_{\mathbf{c},k}(u) = \Psi_{\mathbf{c},k}(u_0)$ for some $0 < u_0 < 1$, then the upper bound in (5.1.3) is attained in the limit by the sequence of parent distribution functions $F_m = u_0 F_{m,a} + (1-u_0)F_{m,b}$, $m = 1, 2, \dots$, where $F_{m,a}$ denote the distribution function of the uniform random variable on the interval $\left[a - \frac{1}{m}, a\right]$, and $a < b$ are arbitrary.
(ii) If $\sup_{0<u<1} \Psi_{\mathbf{c},k}(u) = \lim_{u \searrow 0} \Psi_{\mathbf{c},k}(u)$, then the equality in (5.1.3) is attained in the limit by any sequence of distribution functions $F_m = u_m F_{m,a} + (1-u_m)F_{m,b}$ as $m \to \infty$ and $u_m \searrow 0$, whereas $a < b$.
(iii) If $\sup_{0<u<1} \Psi_{\mathbf{c},k}(u) = \lim_{u \nearrow 1} \Psi_{\mathbf{c},k}(u)$, then the upper bound in (5.1.3) is attained in the limit by any sequence of distribution functions $F_m = u_m F_{m,a} + (1-u_m)F_{m,b}$ as $m \to \infty$ and $u_m \nearrow 1$, with $a < b$.

If $k \ge 2$, then assumption $\mathbb{E}R_{n,k}^2 < \infty$ follows from finiteness of $\mathbb{E}X_1^2$.

Proof. Assume that $X_1, X_2, \ldots$ are i.i.d. with distribution function F and finite variance. Noting that the supports of record values are contained in the supports of the original variables, and using (1.3.2), (1.3.3) and (1.3.6), we conclude

$$\begin{aligned}
& \mathbb{V}ar\left(\sum_{i=1}^{n} c_i R_{i,k}\right) = \sum_{i=1}^{n} c_i^2 \mathbb{V}ar\,(R_{i,k}) + 2\sum_{1\le i<j\le n} c_i c_j \mathbb{C}ov(R_{i,k}, R_{j,k}) \\
= \;& 2\sum_{i=1}^{n} c_i^2 \iint_{0<F(x)\le F(y)<1} \big[F_{i,k}^U\big(F(x)\big) - F_{i,k}^U\big(F(x)\big)F_{i,k}^U\big(F(y)\big)\big]dxdy \\
+ \;& 2\sum_{1\le i<j\le n} c_i c_j \iint_{0<F(x),F(y)<1} \big[F_{i,j,k}^U\big(F(x),F(y)\big) - F_{i,k}^U\big(F(x)\big)F_{j,k}^U\big(F(y)\big)\big]dxdy \\
= \;& 2\iint_{0<F(x)\le F(y)<1} \left\{\sum_{i=1}^{n} c_i^2 F_{i,k}^U\big(F(x)\big)\big[1 - F_{i,k}^U\big(F(y)\big)\big]\right. \\
+ \;& \sum_{1\le i<j\le n} c_i c_j \Bigg\{F_{i,k}^U\big(F(x)\big)\big[1 - F_{j,k}^U\big(F(y)\big)\big] \\
+ \;& F_{j,k}^U\big(F(x)\big)\big[1 - F_{i,k}^U\big(F(y)\big)\big] - [1-F(y)]^k \\
\times \;& \left.\sum_{p=0}^{j-i-1}\sum_{q=0}^{p} \frac{(-1)^q\big[-k\ln\big(1-F(x)\big)\big]^{i+q}\big[-k\ln\big(1-F(y)\big)\big]^{p-q}}{(i-1)!q!(p-q)!(i+q)}\Bigg\}\right\} dxdy \\
= \;& 2\iint_{0<F(x)\le F(y)<1} \left\{\left[\sum_{i=1}^{n} c_i F_{i,k}^U\big(F(x)\big)\right]\left[\sum_{j=1}^{n} c_j\big[1 - F_{j,k}^U\big(F(y)\big)\big]\right] - \sum_{1\le i<j\le n} c_i c_j\right. \\
\times \;& \left.[1-F(y)]^k \sum_{p=0}^{j-i-1}\sum_{q=0}^{p} \frac{(-1)^q\big[-k\ln\big(1-F(x)\big)\big]^{i+q}\big[-k\ln\big(1-F(y)\big)\big]^{p-q}}{(i-1)!q!(p-q)!(i+q)}\right\} dxdy.
\end{aligned}$$

Since

$$\sum_{i=1}^{n} c_i\big[1 - F_{i,k}^U\big(F(x)\big)\big] = [1-F(x)]^k \sum_{i=0}^{n-1}\left(\sum_{j=i+1}^{n} c_j\right)\frac{\big[-k\ln\big(1-F(x)\big)\big]^i}{i!}$$

we have

$$\begin{aligned}
\mathbb{V}ar\left(\sum_{i=1}^{n} c_i R_{i,k}\right) &= 2\iint_{0<F(x)\le F(y)<1} \Phi_{\mathbf{c},k}\big(F(x),F(y)\big)F(x)[1-F(y)]dxdy \\
&\le \sup_{0<u\le v<1} \Phi_{\mathbf{c},k}(u,v)\mathbb{V}ar\,X_1. \qquad (5.1.5)
\end{aligned}$$

Suppose now that (5.1.4) holds.

(i) Assume first that the supremum of $\Psi_{\mathbf{c},k}(u)$ is attained at some $0 < u_0 < 1$. If u_0 is the unique value of $F(x)$, different from 0 and 1, then equality holds in (5.1.5). The condition is satisfied by the distribution functions

$$F(x) = \begin{cases} 0, & x < a, \\ u_0, & a \leq x < b, \\ 1, & x > b \end{cases}$$

for arbitrary $a < b$. Mixtures of uniform distribution functions $F_m(x) = u_0 F_{m,a}(x) + (1 - u_0)F_{m,b}(x)$ tend to $F(x)$ for all $x \in \mathbb{R}$, and function $\Phi_{\mathbf{c},k}(u,v)u(1-v)$ is continuous. Therefore, as $m \to \infty$, we get

$$\begin{aligned} \frac{\mathbb{V}ar_m\left(\sum_{i=1}^n c_i R_{i,k}\right)}{\mathbb{V}ar_m X_1} &= \frac{2\iint_{a-\frac{1}{m}<x<y<b} \Phi_{\mathbf{c},k}(F_m(x), F_m(y))F_m(x)[1-F_m(y)]dxdy}{2\iint_{a-\frac{1}{m}<x<y<b} F_m(x)[1-F_m(y)]dxdy} \\ &\to \frac{2\iint_{a<x<y<b} \Phi_{\mathbf{c},k}(F(x), F(y))F(x)[1-F(y)]dxdy}{2\iint_{a<x<y<b} F(x)[1-F(y)]dxdy} \\ &= \Psi_{\mathbf{c},k}(u_0) = \sup_{0<u<1} \Psi_{\mathbf{c},k}(u). \end{aligned}$$

(ii) If $\sup_{0<u<1} \Psi_{\mathbf{c},k}(u) = \lim_{u\searrow 0} \Psi_{\mathbf{c},k}(u)$, then by the previous statement and continuity of $\Psi_{\mathbf{c},k}$, for the sequence of mixtures defined in Theorem 6 (ii) yields

$$\frac{\mathbb{V}ar_m\left(\sum_{i=1}^n c_i R_{i,k}\right)}{\mathbb{V}ar_m X_1} \to \lim_{u\searrow 0} \Psi_{\mathbf{c},k}(u).$$

The proof of statement (iii) is similar. □

Theorem 7. *Under assumptions of Theorem 6, if either $c_1 = 0$ or $k \geq 2$, then the trivial bound*

$$\frac{\mathbb{V}ar(\sum_{i=1}^n c_i R_{i,k})}{\mathbb{V}ar\, X_1} \geq 0$$

is optimal. If the former (latter) condition holds, then the zero bound is attained for the sequence of baseline distributions described in Theorem 6(ii) (Theorem 6(iii), respectively).

Proof. We first calculate the right limit of (5.1.2) at 0. Denote the respective expression in the curly brackets by $\chi_{\mathbf{c},k}(u)$. Since factor $(1-u)^{k-1}$ is immaterial here, and $\lim_{u\searrow 0} \chi_{\mathbf{c},k}(u) = 0$, with use the l'Hospital rule we obtain

$$\begin{aligned}\lim_{u\searrow 0}\Psi_{\mathbf{c},k}(u) &= \lim_{u\searrow 0}\chi'_{\mathbf{c},k}(u) = k\left(\sum_{j=2}^{n}c_j\right)^2 + k\left(\sum_{j=1}^{n}c_j\right)^2 - 2k\left(\sum_{j=1}^{n}c_j\right)\left(\sum_{j=2}^{n}c_j\right)\\ &= k\left(\sum_{j=1}^{n}c_j - \sum_{j=2}^{n}c_j\right)^2 = kc_1^2.\end{aligned}$$

Hence $c_1 = 0$ implies that $\lim_{u\searrow 0}\Psi_{\mathbf{c},k}(u) = 0$, and using the sequence of baseline distributions of Theorem 6(ii), we attain zero for the variance ratios in the limit. If $k \geq 2$, then $\lim_{u\nearrow 1}\Psi_{\mathbf{c},k}(u) = 0$, because each expression $(1-u)^p[-k\ln(1-u)]^q$ for $p \geq 1$ and $q \geq 0$ tends to 0 as u approaches 1. If $k = 1$, then $\Psi_{\mathbf{c},k}$ tends to $+\infty$ at 1, because it behaves asymptotically as a combination of functions $[-k\ln(1-u)]^i$, $i = 0, \dots, n-1$, and the coefficient $\frac{c_n^2}{(n-1)!}$ of the fastest increasing term $[-k\ln(1-u)]^{n-1}$ is clearly positive. In conclusion, applying construction of Theorem 6(iii) for $k \geq 2$ we obtain zero limit. □

5.2 Single kth record values

In the case of single nth value of kth records with $\mathbf{c} = \mathbf{c}(n) = (0, \dots, 0, 1)$, function (5.1.1) simplifies to

$$\Phi_{\mathbf{c}(n),k}(u,v) = \frac{1-(1-u)^k\sum_{i=0}^{n-1}\frac{[-k\ln(1-u)]^i}{i!}}{u}(1-v)^{k-1}\sum_{i=0}^{n-1}\frac{[-k\ln(1-v)]^i}{i!}.$$

It can be shown that for every $n, k \in \mathbb{N}$ and $\mathbf{c} = \mathbf{c}(n)$, (5.1.4) holds which means that $\sup_{0<u<1}\Psi_{\mathbf{c}(n),k}(u)$ provide the sharp upper bounds. For $n = 1$, we have $R_{1,k} = X_{1:k}$, and results of Section 3.2 apply. When $k = 1$ and $n \geq 2$, we have

$$\sup_{0<u<1}\Psi_{\mathbf{c}(n),k}(u) = \lim_{u\nearrow 1}\Psi_{\mathbf{c}(n),k}(u) = +\infty,$$

and respective attainability conditions are formulated in Theorem 6(iii). Klimczak and Rychlik (2004) showed that for any $n, k \geq 2$ function $\Psi_{\mathbf{c}(n),k}$ is maximized at some inner point $u_0 = u_0(n,k)$ of open interval $(0,1)$. In consequence, the bounds are attained by the approximations of two-point distributions described in Theorem 6(i). Moreover, under the restriction $2 \leq k \leq \max\left\{2, n\frac{n+4}{3n+4}\right\}$, Jasiński (2016) proved that u_0 is the unique zero of $\Psi'_{\mathbf{c}(n),k}$ in $(0,1)$.

Since

$$\begin{aligned}\lim_{u\searrow 0}\Psi_{n,k}(u) &= 0, \qquad n \geq 2,\\ \lim_{u\nearrow 1}\Psi_{n,k}(u) &= 0, \qquad k \geq 2,\end{aligned}$$

the trivial bound

$$\frac{\mathbb{V}ar(\sum_{i=1}^{n} c_i R_{i,k})}{\mathbb{V}ar\, X_1} \geq 0$$

is sharp for all k and n except for $n = k = 1$. For $n \geq 2$ and $k \geq 2$, equality conditions can be found in Theorem 6(ii) and (iii), respectively.

5.3 kth record spacings

In this Section, we thoroughly analyze variances of kth record spacings. To simplify the notation, we write $\Phi_{\mathbf{c}(m+1)-\mathbf{c}(m),k}$ and $\Psi_{\mathbf{c}(m+1)-\mathbf{c}(m),k}$ as $\Phi_{m,k}$ and $\Psi_{m,k}$, respectively. The former has the representation

$$\Phi_{m,k}(u,v) = \frac{[-k\ln(1-u)]^m(1-v)^{k-1}}{um!}\left[1 - \frac{(1-u)^k[-k\ln(1-v)]^m}{m!}\right], \tag{5.3.1}$$

and the latter satisfies $\Psi_{m,k}(u) = \Phi_{m,k}(u,u)$.

Proposition 14. *Let $X_1, X_2, \ldots$ be i.i.d. continuously distributed, and assume that $\mathbb{E}X_1^2 < \infty$ and $\mathbb{E}R_{m+1,k}^2 < \infty$. Then*

$$\frac{\mathbb{V}ar(R_{m+1,k} - R_{m,k})}{\mathbb{V}ar\, X_1} \leq \sup_{0<u<1} \Psi_{m,k}(u), \tag{5.3.2}$$

and the bound is sharp. In particular, the following yields.
(i) If $k = 1$ and $m \geq 1$, then

$$\frac{\mathbb{V}ar(R_{m+1,1} - R_{m,1})}{\mathbb{V}ar\, X_1} \leq \lim_{u \nearrow 1} \Psi_{m,1}(u) = +\infty, \tag{5.3.3}$$

and this upper bound is attained by the sequence of baseline distributions described in Theorem 6(iii).
(ii) If $m = 1$ and $k \geq 2$, then

$$\frac{\mathbb{V}ar(R_{2,k} - R_{1,k})}{\mathbb{V}ar\, X_1} \leq \lim_{u \searrow 0} \Psi_{1,k}(u) = k,$$

and the bound is attained by the sequence of distributions described in Theorem 6(ii).
(iii) If either $k = 2 \leq m$ or $k \geq 3$ with $2 \leq m \leq \frac{2}{3}k$, then

$$\frac{\mathbb{V}ar(R_{m+1,k} - R_{m,k})}{\mathbb{V}ar\, X_1} \leq \Psi_{m,k}(u_0),$$

where $u_0 = u_0(m,k)$ is the unique solution to $\Psi'_{m,k}(u) = 0$, and the equality is attained by the sequence of parent distributions described in Theorem 6(i).
(iv) If finally $k \geq 3$ with $m > \frac{2}{3}k$, then

$$\frac{\mathbb{V}ar(R_{m+1,k} - R_{m,k})}{\mathbb{V}ar\, X_1} \leq \Psi_{m,k}(u_0),$$

where $0 < u_0 < 1$ is the global maximum point of $\Psi_{m,k}$ over $(0,1)$, and attainability conditions are presented in Theorem 6(i).

Remark 9. *Relation (5.3.3) is not surprising in view of the fact that one can construct parent distributions such that $\mathbb{V}ar\, R_{m,1} < \infty = \mathbb{V}ar\, R_{m+1,1}$ for every $m \in \mathbb{N}$ (cf Nagaraja, 1978, and Klimczak and Rychlik, 2004). It is surprising, though, that that arbitrary large values of variance ratio are possible for so simple parent distributions with very restricted supports, defined in Theorem 6.*

Remark 10. *In the last case of Proposition 14 we can restrict the number of local maxima of $\Psi_{m,k}$ with use of the variation diminishing property of power series of Lemma 4. We can check that the derivative $\Psi'_{m,k}(u)$, $0 < u < 1$, has for arbitrary $k, m \geq 3$ at most 5 zeros, and, in consequence, $\Psi_{m,k}(u)$ itself has 3 local maxima at most. Moreover, under some extra conditions we are able to restrict the number of local maxima to two. We provide the respective arguments just below the proof of Proposition 14.*

On the other hand, many numerical examples show that for various $k, m \geq 3$ function $\Psi_{m,k}(u)$ is merely increasing-decreasing in $(0,1)$, and has a single maximum there. However, we are not able to prove the claim formally.

In the proof of Proposition 14, we use Lemma 7, and Lemma 9 below which follows from the following one.

Lemma 8. *Let*

$$f_{n,M}(x) = e^x - M\frac{x^n}{n!}$$

be defined on the non-negative half-axis for fixed real M and non-negative integer n. Then $f_{0,M}(0) = 1 - M$ and $f_{n,M}(0) = 1$ for $n \in \mathbb{N}$. Also, $\lim_{x \nearrow \infty} f_{n,M}(x) = +\infty$ for every $n = 0, 1, \ldots$ and $M \in \mathbb{R}$. Moreover, we have the following.
(i) Function $f_{0,M}$ is strictly increasing everywhere.
(ii) If $M \leq 1$, then $f_{1,M}$ also increases, and otherwise there exists $x_0 > 0$ such that $f_{1,M}$ decreases on $(0, x_0)$ and increases on (x_0, ∞).
(iii) Let $n \geq 2$. If either $M \leq 1$ or $M > 1$ and $\min_{x>0} f_{n-1,M}(x) \geq 0$, then $f_{n,M}$ is increasing. If $M > 1$ and $\min_{x>0} f_{n-1,M}(x) < 0$, then there exist $0 < x_1 < x_2 < \infty$ such that $f_{n,M}$ increases on $(0, x_1)$, decreases on (x_1, x_2), and finally increases on (x_2, ∞).

Proof. Calculating the left- and right-end values is immediate. Also, claim (i) is trivial since $f'_{0,M}(x) = e^x > 0$, $x > 0$. Note that $f'_{n+1,M} = f_{n,M}$ for all $n \in \mathbb{N}$.
(ii) We have $f'_{1,M}(x) = e^x - M$. If $M \le 1$, then $f_{1,M}$ is strictly increasing and positive. Otherwise, it is decreasing on $(0, x_0)$ with $x_0 = x_0(1) = \ln M$, and increasing elsewhere.
(iii) Suppose first that $n = 2$. If $M \le 1$, then $f'_{2,M} = f_{1,M} > 0$, and, due to (ii), $f_{2,M}$ is increasing from 1 to ∞. If $M > 1$ and $\min_{x>0} f_{1,M}(x) = f_{1,M}(\ln M) \ge 0$, then by the latter statement of (ii), $f_{2,M}$ is increasing as well. For $M > 1$ with $f_{1,M}(\ln M) < 0$, by the same argument, there exist $0 < x_1 = x_1(2) < x_0(1) < x_2 = x_2(2)$ such that $f'_{2,M}(x_i) = f_{1,M}(x_i) = 0$, $i = 1, 2$, and, consequently, $f_{2,M}$ is increasing on $(0, x_1) \cup (x_2, \infty)$, and decreasing on (x_1, x_2). It means that conditions of (iii) are satisfied for $n = 2$.

Assume now that (iii) holds for some $n \ge 2$. We conclude the same for $n + 1$. If $M \le 1$, then $f'_{n+1,M}(x) = f_{n,M}(x)$ increases on $\mathbb{R}_+$ from 1 to ∞, and so does $f_{n+1,M}$. If $M > 1$ and $\min_{x>0} f_{n,M}(x) = f_{n,M}(x_0(n)) \ge 0$, then $f'_{n+1,M}(x)$ is positive (except for possibly at $x_0(n)$ when $f_{n,M}(x_0(n)) = 0$), and so $f_{n+1,M}$ is increasing function. If finally $M > 1$ and $f_{n,M}(x_0(n)) < 0$, then there are $0 < x_1 = x_1(n+1) < x_0(n) < x_2 = x_2(n+1) < \infty$ such that $f_{n+1,m}$ is increasing, decreasing, and increasing in $(0, x_1)$, (x_1, x_2), and (x_2, ∞), respectively. □

Lemma 9. *Let f and g be polynomials, $r(x) = \frac{g(x)}{f(x)}$, $n \in \mathbb{N}$, and*

$$h(x) = f(x)e^x - g(x)\frac{x^n}{n!}, \qquad x \ge 0.$$

(i) If $\lim_{x\to\infty} f(x) = \infty$, $x_f = \max\{x \in \mathbb{R}\colon f(x) = 0\} \ge n$, $g(x_f) > 0$, and $r'(x) < 0$ for $x > x_f$, then there exists $x_h > x_f$ such that h is negative on (x_f, x_h), and positive on (x_h, ∞).

Moreover, function $h(x)$ is increasing on (x_h, ∞) under additional assumption that $f'(x) > 0$ for $x > x_h$.
(ii) If $f(0) > 0$, $0 < x_f = \min\{x \in \mathbb{R}\colon f(x) = 0\} \le n$, $g(x_f) > 0$, and $r'(x) > 0$ for $0 < x < x_f$, then there exists $0 < x_h < x_f$ such that h is positive on $(0, x_h)$, and negative on (x_h, x_f).

Proof. (i)Note that $h(x_f) = -g(x_f)\frac{x_f^n}{n!} < 0$ and $\lim_{x\to\infty} h(x) = \infty$. Hence the function has some zeros in (x_f, ∞). Let x_h denote the smallest of them. Obviously, h is negative on (x_f, x_h).

Define now $\varphi(x) = \frac{h(x)}{f(x)} = e^x - r(x)\frac{x^n}{n!}$, $x > x_f$, and $\psi(x) = e^x - M\frac{x^n}{n!}$ with $M = r(x_h)$. Note that $f(x_h) > 0$, and $\psi(x_h) = \varphi(x_h) = \frac{h(x_h)}{f(x_h)} = 0$. By assumption, $r(x) < r(x_h) = M$, $x > x_h > x_f$, and so

$$\varphi(x) = e^x - r(x)\frac{x^n}{n!} > e^x - M\frac{x^n}{n!} = \psi(x), \qquad x > x_h.$$

Observe further that

$$\psi'(x_h) = e^{x_h} - M\frac{x_h^{n-1}}{(n-1)!} > 0 = \psi(x_h) = e^{x_h} - M\frac{x_h^n}{n!}$$

iff $M < M\frac{x_h}{n}$. If $M \leq 1$, then by Lemma 8 (ii) and (iii) ψ increases from 1 at 0 to ∞ at ∞, which contradicts $\psi(x_h) = 0$. Therefore condition $\psi'(x_h) > 0$ is equivalent with $x_h > n$ which is true due to inequalities $n \leq x_f < x_h$. By Lemma 8 again, relations $0 = \psi(x_h) < \psi'(x_h)$ imply that $\psi(x) > 0$ for all $x > x_h$, and so is $\varphi(x) = \frac{h(x)}{f(x)}$. By positivity of f on (x_f, ∞), $h(x) > 0$ for $x > x_h$ as well.

Furthermore, note that for $x > x_h$ yields

$$\begin{aligned} \varphi'(x) &= e^x - r'(x)\frac{x^n}{n!} - r(x)\frac{x^{n-1}}{(n-1)!} > e^x - r(x)\frac{x^{n-1}}{(n-1)!} \\ &> e^x - r(x_h)\frac{x^{n-1}}{(n-1)!} = \psi'(x) > 0, \end{aligned}$$

and consequently, under condition $f'(x) > 0$, $x > x_h$, we have $h'(x) = f'(x)\varphi(x) + f(x)\varphi'(x) > 0$, $x > x_h$, as well.

(ii) The proof is based on similar ideas. Since $h(x_f) = -g(x_f)\frac{x_f^n}{n!} < 0 < f(0) = h(0)$, and h is continuous, it has some zeros in $(0, x_f)$. Let x_h be the greatest of them. Clearly, $h < 0$ on (x_h, x_f). Functions $\varphi(x) = e^x - r(x)\frac{x^n}{n!}$ and $\psi(x) = e^x - M\frac{x^n}{n!}$ with $M = r(x_h)$ vanish at x_h. Inequality $\varphi(x) > \psi(x)$, $0 < x < x_h$, follows from condition $r'(x) > 0$, $0 < x < x_f$. Relation $M \leq 1$ implying positive increase of ψ on $\mathbb{R}_+$ contradicts $\psi(x_h) = 0$. When $M > 1$, conditions $x_h < x_f \leq n$ imply $\psi'(x_h) < \psi(x_h) = 0$. Due to Lemma 8 (ii) and (iii), and the last relations, x_h is the first zero of ψ, and this is positive on $(0, x_h)$. Therefore so is $\varphi(x) = \frac{h(x)}{f(x)} > \psi(x)$ there. Positivity of h on $(0, x_h)$ follows from positivity of f in $(0, x_f) \supset (0, x_h)$. □

Proof of Proposition 14. We start with proving that

$$\Phi_{m,k}(u, v) > \Phi_{m,k}(u, u) = \Psi_{m,k}(u), \qquad u < v < 1,$$

for every fixed $0 < u < 1$. The two-variable function, defined in (5.3.1), is represented as the product of two factors. The first one is obviously non-increasing in v (constant for $k = 1$ and decreasing otherwise), and positive for all $u < v < 1$. The latter is also decreasing. In order to show its positivity, we note that

$$1 - (1-u)^k \frac{[-k\ln(1-u)]^m}{m!} > 1 - (1-u)^k \sum_{i=0}^{m} \frac{[-k\ln(1-u)]^i}{i!} = F^U_{m+1,k}(u) > 0,$$

where $F^U_{m+1,k}$ denotes the distribution function of the $(m+1)$st value of the kth record based on i.i.d. variables with the standard uniform distribution. The product of positive non-increasing and positive decreasing functions is positive decreasing. It follows that

$$\sup_{0<u\leq v<1} \Phi_{m,k}(u,v) = \sup_{0<u<1} \Psi_{m,k}(u).$$

By Theorem 6, the upper bound (5.3.2) is sharp. In order to determine its attainability conditions, it suffices to find the arguments for which functions $\Psi_{m,k}$ attain their suprema.

(i) Assume first that $k=1$. Then

$$\Psi_{m,1}(u) = \frac{[-\ln(1-u)]^m}{um!}\left[1 - \frac{(1-u)[-\ln(1-u)]^m}{m!}\right].$$

As $u \nearrow 1$, the first term increases to $+\infty$, whereas the other one tends to 1. Therefore the upper bound trivially equals to $+\infty$. Since

$$\sup_{0<u<1} \Psi_{m,k}(u) = \lim_{u\nearrow 1} \Psi_{m,k}(u),$$

the attainability conditions are presented in Theorem 6(iii).

(ii) We start with analyzing case $m=1$ with $k=2$. Then

$$\Psi_{1,2}(u) = \frac{-2(1-u)\ln(1-u)]}{u}\left[1+2(1-u)^2\ln(1-u)\right] = f(u)g(u), \quad 0<u<1,$$

say. We have $f'(u) = \frac{2}{u^2}h(u)$ with $h(u) = u+\ln(1-u)$. Since $h(0)=0$ and $h'(u) = -\frac{u}{1-u} < 0$, derivative f' is decreasing and negative. Function f is decreasing as well, and positive with $\lim_{u\searrow 0} f(u) = 2$ and $f(1)=0$. The latter function has derivative $g'(u) = 2(1-u)[-2\ln(1-u)-1]$, and so is decreasing in $(0, 1-e^{-1/2})$ and increasing in $(1-e^{-1/2}, 1)$. Its maximums are $g(0) = g(1) = 1$. Therefore

$$\sup_{0<u<1} \Psi_{1,2}(u) = \lim_{u\searrow 0} \Psi_{1,2}(u) = \lim_{u\searrow 0} f(u)g(u) = 2.$$

This proves (ii) for $k=2$.

For the other cases $m=1$ with $k\geq 3$, and $m\geq 2$ with $k\geq 2$, we first analyze functions $\psi_{m,k}(u) = u\Psi_{m,k}(u)$. Then we obviously have $\lim_{u\searrow 0}\psi_{m,k}(u) = \lim_{u\nearrow 1}\psi_{m,k}(u) = 0$. We first show that each $\psi_{m,k}$ is first increasing and then decreasing in $(0,1)$. We write down the derivative

$$\begin{aligned}\psi'_{m,k}(u) &= (1-u)^{2k-2}\frac{\left[-k\ln(1-u)\right]^{m-1}}{m!}\left\{\frac{mk-(k-1)\left[-k\ln(1-u)\right]}{(1-u)^k}\right.\\ &- \left.\frac{[-k\ln(1-u)]^m\left[2mk-(2k-1)[-k\ln(1-u)]\right]}{m!}\right\}.\end{aligned} \tag{5.3.4}$$

Consider the strictly increasing variable transformation $x : (0,1) \mapsto \mathbb{R}_+$ defined as $x(u) = -k\ln(1-u)$, with the inverse $u(x) = 1 - \exp(-x/k)$. We have

$$\chi_{1,m,k}(x) = \psi'_{m,k}(u(x)) = -\frac{x^{m-1}}{m!}\exp\left(-\frac{2(k-1)x}{k}\right)\left[f_{1,m,k}(x)e^x - g_{1,m,k}(x)\frac{x^m}{m!}\right], \qquad (5.3.5)$$

where

$$f_{1,m,k}(x) = (k-1)x - km, \qquad (5.3.6)$$

$$g_{1,m,k}(x) = (2k-1)x - 2km, \qquad x > 0, \qquad (5.3.7)$$

We denote the expression in the brackets of (5.3.5) by $h_{1,m,k}(x)$. We now prove that function $h_{1,m,k}(x)$, $x > 0$, is first negative and then positive. This would imply that (5.3.5) and (5.3.4) are first positive, and ultimately negative in their domains, as desired. Functions (5.3.6) and (5.3.7) are initially negative and then positive, and their zeros are $x_f(m,k) = \frac{k}{k-1}m$ and $x_g(m,k) = \frac{2k}{2k-1}m$, respectively. Notice that $m < x_g(m,k) < x_f(m,k)$. We first show that $h_{1,m,k}$ is negative in $(0, x_g(m,k))$. To this end, we use the relations

$$f_{1,m,k}(x)e^x < f_{1,m,k}(x)\frac{x^{m-1}}{(m-1)!}\left[1 + \frac{x}{m}\right] < g_{1,m,k}(x)\frac{x^m}{m!}.$$

The former is trivial by negativity of $f_{1,m,k}$ on $(0, x_g(m,k))$, and the latter is equivalent to inequality $kx^2 - (2k-1)m + km^2 > 0$. This is true for all $k, m \in \mathbb{N}$ and $x \in \mathbb{R}$, because its discriminant $\Delta = -m^2(4k-1)$ is negative then. Function $h_{1,m,k}$ is negative in $(x_g(m,k), x_f(m,k))$ as well, because both its summands are negative there. Finally, since

$$\frac{d}{dx}\frac{g_{1,m,k}(x)}{f_{1,m,k}(x)} = \frac{-km}{[(k-1)x - km]^2} < 0,$$

and (5.3.6) and (5.3.7) satisfy all other assumptions of Lemma 9(i), we deduce that $h_{1,m,k}$ is first negative, and then positive in $(x_f(m,k), +\infty)$. This completes the proof that $\psi_{m,k}$ is first increasing and then decreasing in $(0,1)$.

(ii) cont. Suppose now that $m = 1$ and $k \geq 3$. Then

$$\begin{aligned}\psi''_{1,k}(u) &= (1-u)^{2k-3}\{-k(2k-3)(1-u)^{-k} + (k-1)(k-2)[-k\ln(1-u)](1-u)^{-k} \\ &- 2k^2 + 2k(4k-3)[-k\ln(1-u)] - (2k-1)(2k-2)[-k\ln(1-u)]^2\}.\end{aligned}$$

Using again change of variable $x(u) = -k\ln(1-u)$, we obtain

$$\begin{aligned}\chi_{2,1,k}(x) &= \exp\left(-\frac{(2k-3)x}{k}\right)h_{2,1,k}(x) \\ &= \exp\left(-\frac{(2k-3)x}{k}\right)[e^x f_{2,1,k}(x) - g_{2,m,k}(x)x - 2k^2]\end{aligned}$$

with

$$f_{2,1,k}(x) = (k-1)(k-2)x - k(2k-3), \tag{5.3.8}$$
$$g_{2,1,k}(x) = (2k-1)(2k-2)x - 2k(4k-3).$$

Function (5.3.8) has a single zero at $x_f(1,k) = \frac{k(2k-3)}{(k-1)(k-2)}$. We intend to prove that $h_{2,1,k}$ is negative in $(0, x_f(1,k))$, and changes the sign once in $(x_f(1,k), \infty)$ from minus to plus. To verify the first claim we notice that

$$h_{2,1,k}(x) < A_k(x) = \left(1 + x + \frac{x^2}{2}\right) f_{2,1,k}(x) - g_{2,1,k}(x)x - 2k^2$$

for $x \in (0, x_f(1,k))$, because $f_{2,1,k}$ is negative there. We claim that the cubic function

$$\begin{aligned} A_k(x) &= \frac{1}{2}(k-1)(k-2)x^3 - k\left(4k - \frac{9}{2}\right)x^2 + (7k^2 - 6k + 2)x - (4k+3)k \\ &= ax^3 + bx^2 + cx + d, \end{aligned}$$

is negative for $0 < x < x_f(1,k)$. Since $A_k(x_f(1,k)) = -\frac{2k^2(k^2+2k-5)}{(k-2)^2}$ is negative for all $k \geq 3$, it suffices to show that the discriminant $D = q^2 + p^3$ of A_k is positive, where $3p = \frac{3ac-b^2}{3a^2}$, $2q = \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}$. Elementary calculations show that $D = \frac{g(k)}{27(k-1)^4(k-2)^4}$, where

$$g(k) = 104k^8 + 120k^7 - 1584k^6 + 132k^5 + 6353k^4 - 9216k^3 + 5296k^2 - 1344k + 128.$$

We also introduce $g1(k) = g(k+1) - g(k)$ and $g2(k) = g1(k+1) - g1(k)$ which amount to

$$\begin{aligned} g1(k) &= 832k^7 + 3752k^6 - 1160k^5 - 11620k^4 + 5076k^3 - 6538k^2 + 1184k - 139, \\ g2(k) &= 5824k^6 + 39984k^5 + 79600k^4 + 46080k^3 + 7660k^2 - 21792k - 8474, \end{aligned}$$

respectively. The derivative of $g2$ is increasing for positive k, because it is a positive combination of power functions and a negative constant. Due to $g2'(3) = 34550040$, it is also positive for $k \geq 3$. Since all $g2(3) = 21648658$, $g1(3) = 3413315$, and $g(3) = 131645$ are positive, we successively conclude that $g2$, $g1$, and finally g itself are positive for all $k \geq 3$. This is the desired statement.

We now check that function

$$\tilde{h}_{2,1,k}(x) = h_{2,1,k}(x) + 2k^2 = e^x f_{2,1,k}(x) - g_{2,1,k}(x)x$$

fulfills conditions of Lemma 9(i). Indeed, $f_{2,1,k}(x) \nearrow \infty$ as $x \nearrow \infty$, $x_f(1,k) = \frac{k(2k-3)}{(k-1)(k-2)} > n = 1$, $g_{2,1,k}\left(\frac{k(2k-3)}{(k-1)(k-2)}\right) = \frac{6k(k-1)}{k-2} > 0$, and

$$\frac{d}{dx}\frac{g_{2,1,k}(x)}{f_{2,1,k}(x)} = \frac{-6k(k-1)^2}{[(k-1)(k-2)x - k(2k-3)]^2} < 0,$$

when $k \geq 3$. So there exists $\tilde{x}_{\tilde{h}}(1,k) > x_f(1,k)$ such that $\tilde{h}_{2,1,k}(x)$ is negative for $x_f(1,k) < x < \tilde{x}_{\tilde{h}}(1,k)$ and positive for $x > \tilde{x}_{\tilde{h}}(1,k)$, and it tends to $+\infty$ as x increases to $+\infty$. Since $f'_{2,1,k}(x) > 0$, by the last claim of Lemma 9(i), $\tilde{h}_{2,1,k}$ is also increasing for $x > \tilde{x}_{\tilde{h}}(1,k)$. Accordingly, function $h_{2,1,k} = \tilde{h}_{2,1,k} - 2k^2$ which is negative on $(x_f(1,k), \tilde{x}_{\tilde{h}}(1,k))$, is increasing to ∞ on $(\tilde{x}_{\tilde{h}}(1,k), \infty)$ as well. Hence it is negative on $(x_f(1,k), x_h(1,k))$ and positive on $(x_h(1,k), \infty)$ for some $x_h(1,k) > \tilde{x}_{\tilde{h}}(1,k)$. This implies that $\psi''_{1,k}$ is first negative and then positive on $(0,1)$.

Recall that $\psi'_{1,k}$ is consecutively positive and negative. Since every smooth function is concave around its local maximum, $\psi_{1,k}$ that starts from 0 at 0 is first concave increasing, then concave decreasing, and finally convex decreasing and vanishes at the right-end point 1. By Lemma 7(iv)(a) and (i), respectively, $\Psi_{1,k}$ decreases on whole $(0,1)$. This ends the proof of Proposition 14(ii).

(iii) For $m \geq 2$ and $k \geq 2$, the second derivative has the form

$$\begin{aligned}
\psi''_{m,k}(u) &= (1-u)^{2k-3}\frac{[-k\ln(1-u)]^{m-2}}{m!}\Bigg[-(2k-3)km[-k\ln(1-u)](1-u)^{-k} \\
&+ (k-1)(k-2)[-k\ln(1-u)]^2(1-u)^{-k} - 2k^2m(2m-1)\frac{[-k\ln(1-u)]^m}{m!} \\
&+ k^2m(m-1)(1-u)^{-k} + 2k(4k-3)m\frac{[-k\ln(1-u)]^{m+1}}{m!} \\
&- (2k-1)(2k-2)\frac{[-k\ln(1-u)]^{m+2}}{m!}\Bigg],
\end{aligned}$$

and its transformation is equal to

$$\chi_{2,m,k}(x) = \frac{x^{m-2}}{m!}\exp\left(-\frac{(2k-3)x}{k}\right)h_{2,m,k}(x),$$

where

$$h_{2,m,k}(x) = f_{2,m,k}(x)e^x - g_{2,m,k}(x)\frac{x^m}{m!} \tag{5.3.9}$$

with

$$\begin{aligned}
f_{2,m,k}(x) &= (k-1)(k-2)x^2 - k(2k-3)mx + k^2m(m-1), &(5.3.10)\\
g_{2,m,k}(x) &= (2k-1)(2k-2)x^2 - 2k(4k-3)mx + 2k^2m(2m-1). &(5.3.11)
\end{aligned}$$

We first focus on the case $k = 2$ and $m \geq 2$ with

$$\begin{aligned}
f_{2,m,2}(x) &= -2m(x - 2m + 2), \\
g_{2,m,2}(x) &= 6x^2 - 20mx + 8m(2m-1).
\end{aligned}$$

The first one is linear decreasing with zero at $x_f(m,2) = 2(m-1) > 0$. We start with proving that $h_{2,m,2}(x) > 0$, $0 < x < x_f(m,2)$. Since $e^x f_{2,m,2}(x) > \left[\frac{x^{m-2}}{(m-2)!} + \frac{x^{m-1}}{(m-1)!}\right] f_{2,m,2}(x)$ then, it suffices to show that

$$\left(1 + \frac{x}{m-1}\right) f_{2,m,2}(x) > g_{2,m,2}(x) \frac{x^2}{(m-1)m}, \qquad 0 < x < x_f(m,2).$$

This inequality can be rewritten as

$$p_m(x) = \frac{3x^4 - 10mx^3 + (9m-4)mx^2 - m^2(m-1)x - 2m^2(m-1)^2}{3} < 0$$

for $0 < x < x_f(m,2)$. Polynomial $p_2(x) = x^4 - \frac{20}{3}x^3 + \frac{28}{3}x^2 - \frac{4}{3}x - \frac{8}{3}$ has two real zeros, which are approximately equal to -0.4140 and 4.8066. Therefore this is negative on $(0, x_f(m,2)) = (0,2)$. From now on we take $m \geq 3$, and define a family of polynomials

$$p_{m,c}(x) = x(x-2m)\left(x^2 - \frac{4}{3}mx + c\right), \qquad m \geq 3,\ c > \frac{4}{9}m^2.$$

Under this restriction, the last factor is always positive, and $p_{m,c}$ is negative in $(0, 2m) \supset (0, x_f(m,2))$. For every $m \geq 3$ we find $c_0 = c_0(m) > \frac{4}{9}m^2$ such that

$$p_m(x) < p_{m,c_0}(x) < 0, \qquad 0 < x < x_f(m,2). \tag{5.3.12}$$

Put

$$d_{m,c}(x) = p_{m,c}(x) - p_m(x) = \frac{(3c - m^2 + 4m)x^2 - m(6c - m^2 + m)x + 2m^2(m-1)^2}{3}.$$

This function is convex and has a negative discriminant under the conditions $c > \frac{m(m-4)}{3}$ and

$$\left(\frac{1}{2}m - \frac{1}{3} - \frac{1}{3}\sqrt{5m+1}\right)(m-1) < c < \left(\frac{1}{2}m - \frac{1}{3} + \frac{1}{3}\sqrt{5m+1}\right)(m-1),$$

respectively. The former is satisfied for all $c > \frac{4}{9}m^2$. From the latter one we pick $c_0 = \left(\frac{1}{2}m - \frac{1}{3} + \frac{1}{3}\sqrt{5m}\right)(m-1)$, and check below that this is greater than $\frac{4}{9}m^2$. For $a(m) = c_0(m) - \frac{4}{9}m^2$, we have $a(3) = \frac{2\sqrt{15}-5}{3} \approx 0.9153 > 0$ and

$$a'(m) = \frac{2m\sqrt{m} + 9\sqrt{5}m - 15\sqrt{m} - 3\sqrt{5}}{18\sqrt{m}} = \frac{b(\sqrt{m})}{18\sqrt{m}},$$

say. It remains to observe that $b(\mu) > 0$, $\mu \geq 1$, because $b(1) = 6\sqrt{5} - 13 \approx 0.4164 > 0$, and $b'(\mu) = 3(2\mu^2 + 6\sqrt{5}\mu - 5)$ is increasing and positive for $\mu \geq 1$. In consequence, $a(m) > 0$ for every $m \geq 3$. This completes the proof (5.3.12) and guarantees positivity of $h_{2,m,2}$ on $(0, x_f(m,2))$ for $m \geq 2$.

For $x > x_f(m,2) = 2(m-1)$ we apply Lemma 9(i) for showing that $-h_{2,m,2}(x) = -e^x f_{2,m,2}(x) + g_{2,m,2}(x)\frac{x^m}{m!}$ changes the sign once from $-$ into $+$. Note that $x_f(m,2) = 2(m-1) > m$, $-g_{2,m,2}(2(m-1)) = 16\left(m - \frac{3}{2}\right) > 0$ and

$$\frac{d}{dx}\frac{g_{2,m,2}(x)}{f_{2,m,2}(x)} = -\frac{3x^2 - 12(m-1)x + 4m(3m-4)}{m(x-2m+2)^2} < 0,$$

because the discriminant of the numerator $\Delta = -96\left(m - \frac{3}{2}\right) < 0$ for $m \geq 2$. Accordingly, the claim holds, and we conclude that $h_{2,m,2}$ is first positive, and eventually negative on the positive half-axis.

The sign sequences of all functions $\psi'_{2,m,2}(u)$ and $\psi''_{2,m,2}(u)$, $0 < u < 1$, $m \geq 2$, are plus and minus only. Therefore each $\psi_{2,m,2}$ is consecutively convex increasing, concave increasing and concave decreasing, and always positive. By Lemma 7(iiib), (ivc) and (i), $\Psi_{2,m,2}$ is increasing from 0 at the origin to the maximum located in the interval of concave increase of $\psi_{2,m,2}$, and decreases to 0 at 1. This proves the Proposition 14(iii) for $k = 2$.

Now we prove the analogous statement for $k \geq 3$ and $2 \leq m \leq \frac{2}{3}k$. Observe first that function (5.3.10) has two zeros, because its discriminant $\Delta_f(m,k) = k^2 m[4(k-1)(k-2) + m] > 0$. They amount to

$$x_{f,i}(m,k) = \frac{(2k-3)m + (-1)^i\sqrt{m[4(k-1)(k-2)+m]}}{2(k-1)(k-2)}\,k, \qquad i = 1,2.$$

The latter is clearly positive, and so is the former, because $f_{2,m,k}(0) = k^2 m(m-1) > 0$. We analyze the sign changes of (5.3.9) in the intervals $(0, x_{f,1}(m,k))$, $(x_{f,1}(m,k), x_{f,2}(m,k))$, and $(x_{f,2}(m,k), \infty)$, when $2 \leq m \leq \frac{2}{3}k$.

Firstly we show that $h_{2,m,k}$ is first positive and then negative in the first interval. To this end we check that functions $f = f_{2,m,k}$, $g = g_{2,m,k}$ satisfy the assumptions of Lemma 9(ii) with $x_f = x_{f,1}(m,k)$ and $n = m$. We start with verifying relation $x_{f,1}(m,k) \leq m$ for $m \leq \frac{k^2}{2}$ (note that this condition is weaker than $m \leq \frac{2}{3}k$, because $\frac{2}{3}k \leq \frac{k^2}{2}$ for $k \geq 2$). The inequality can be rewritten as

$$(3k-4)\sqrt{m} \leq k\sqrt{4(k-1)(k-2)+m}.$$

Squaring the left- and right-hand side expressions, we obtain the inequality which simplifies to $4(k^2 - 2m)(k-1)(k-2) > 0$, as desired.

Now we show that $g_{2,m,k}(x_{f,1}(m,k)) > 0$ for $m < \frac{(k+1)^2}{6}$. This restriction implies $m \leq \frac{2}{3}k$, because $\frac{2}{3}k < \frac{(k+1)^2}{6}$ for all $k \neq 1$. We have $g_{2,m,k}(x_{f,1}(m,k)) = \frac{k^2 m}{(k-2)^2} a_1(m,k)$, where

$$a_1(m,k) = 2(k+1)(k-2) + 3m - 3\sqrt{m[4(k-1)(k-2)+m]}. \tag{5.3.13}$$

This is positive when $[2(k+1)(k-2)+3m]^2 > 9m[4(k-1)(k-2)+m]$, and this is equivalent to $4(k-2)^2[(k+1)^2 - 6m] > 0$.

We finally check that $r(x) = \frac{g_{2,m,k}(x)}{f_{2,m,k}(x)}$ has positive derivative in $(0, x_{f,1}(m,k))$, when $2 \leq m \leq \frac{2}{3}k$. We have $r'(x) = \frac{2km}{f^2_{2,m,k}(x)} a_{m,k}(x)$, where

$$a_{m,k}(x) = -3(k-1)^2 x^2 + 2k(k-1)(3m-k-1)x + k^2 m(2k-3m).$$

This is concave, and its discriminant $\Delta_a = 4k^2(k-1)^2[(k+1)^2 - 6m] > 0$ when $m < \frac{(k+1)^2}{6}$. Hence $a_{m,k}$ is positive in $(0, x_{f,1}(m,k))$ (and so is r') if it is non-negative at the end-points. We have $a_{m,k}(0) = k^2 m(2k-3m) \geq 0$ when $m \leq \frac{2}{3}k$. Moreover, $a_{m,k}(x_{f,1}(m,k)) = \frac{k^2}{2(k-2)^2} a_2(m,k)$, where

$$a_2(m,k) = \sqrt{m[4(k-1)(k-2)+m]}\, a_1(m,k).$$

$\big($cf 5.3.13$\big)$. This is non-negative if $m \leq \frac{(k+1)^2}{6}$. Since all the conditions of Lemma 9(ii) are fulfilled, $h_{2,m,k}$ has a single zero in $(0, x_{f,1}(m,k))$, and it changes the sign there from plus to minus.

Next we prove that $h_{2,m,k}(x) < 0$, $x_{f,1}(m,k) < x < x_{f,2}(m,k)$ for all $k \geq 3$ and $2 \leq m \leq \frac{2}{3}k$. We first treat case $m = 2$, $k = 3$. Since $f_{2,2,3}(x) < 0$, $x \in (x_{f,1}(2,3), x_{f,2}(2,3)) = \left(\frac{3}{2}(3-\sqrt{5}), \frac{3}{2}(3+\sqrt{5})\right) \approx (1.1459, 7.8541)$, we have

$$h_{2,2,3}(x) < \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right) f_{2,2,3}(x) - g_{2,2,3}(x)\frac{x^2}{2}.$$

The right-hand side equals to $x^5 - 36x^4 + 150x^3 - 183x^2 + 54$. This polynomial has three zeros approximately equal to -0.4565, 0.7904, 31.4099, and is negative in $(0.7904, 31.4099) \supset (1.1459, 7.8541)$. It remains to consider $k \geq 4$ with $2 \leq m \leq \frac{2}{3}k$, and use the bound

$$h_{2,m,k}(x) < \frac{x^{m-1}}{(m-1)!}\left[1 + \frac{x}{m} + \frac{x^2}{m(m+1)}\right] f_{2,m,k}(x) - g_{2,m,k}(x)\frac{x^m}{m!}.$$

The right hand-side is negative iff

$$\begin{aligned} p_{m,k}(x) &= (k-1)(k-2)x^4 + (-5k^2m - 3k^2 + 6km + 3k)x^3 \\ &+ (8k^2m^2 + 6k^2m - 6km^2 - 6km + 2m^2 + 2m)x^2 \\ &+ (-5k^2m^3 - 4k^2m^2 + 3km^3 + k^2m + 3km^2)x + k^2(m^4 - m^2) < 0. \end{aligned}$$

Define

$$q_{m,k}(x) = (k-1)(k-2)(x-1)^2[x - x_{f,1}(m,k)][x - x_{f,2}(m,k)],$$

which is non-positive in $(x_{f,1}(m,k), x_{f,2}(m,k))$, and set

$$d_{m,k}(x) = q_{m,k}(x) - p_{m,k}(x) = a(m,k)x^3 + b(m,k)x^2 + c(m,k)x + d(m,k),$$

where

$$\begin{aligned}
a(m,k) &= 3k^2m + k^2 - 3km + 3k - 4, \\
b(m,k) &= -7k^2m^2 - 3k^2m + 6km^2 + k^2 - 2m^2 - 3k - 2m + 2, \\
c(m,k) &= 5k^2m^3 + 2k^2m^2 - 3km^3 - k^2m - 3km^2 + 3km, \\
d(m,k) &= -k^2m^4 + 2k^2m^2 - mk^2.
\end{aligned}$$

With use of $p(m,k) = \frac{3a(m,k)c(m,k) - b^2(m,k)}{9a(m,k)^2}$ and $q(m,k) = \frac{2b^3(m,k)}{54a^3(m,k)} - \frac{b(m,k)c(m,k)}{6a^2(m,k)} + \frac{d(m,k)}{2a(m,k)}$, we write down its discriminant

$$D(m,k) = p^3(m,k) + q^2(m,k) = \frac{k^2m}{180a(m,k)^4}\,\delta_0(m,k),$$

where $\delta_0(m,k) = \sum_{i=0}^{6} a_{0,i}(m)k^i$, with

$$\begin{aligned}
a_{0,0}(m) &= -4m^9 - 48m^8 - 136m^7 - 32m^6 + 276m^5 + 128m^4 - 264m^3 - 48m^2 \\
&+ 124m - 32, \\
a_{0,1}(m) &= 48m^9 + 480m^8 + 1452m^7 + 948m^6 - 2112m^5 - 2076m^4 + 1680m^3 \\
&+ 996m^2 - 876m + 144, \\
a_{0,2}(m) &= -196m^9 - 1660m^8 - 5052m^7 - 4708m^6 + 4975m^5 + 8458m^4 \\
&- 3163m^3 - 4302m^2 + 2339m - 264, \\
a_{0,3}(m) &= 288m^9 + 2124m^8 + 6432m^7 + 7782m^6 - 2238m^5 - 11184m^4 - 24m^3 \\
&+ 6360m^2 - 2502m + 252, \\
a_{0,4}(m) &= -64m^9 - 440m^8 - 1905m^7 - 4044m^6 - 2602m^5 + 2818m^4 + 3331m^3 \\
&- 3042m^2 + 923m - 132, \\
a_{0,5}(m) &= -228m^7 - 732m^6 - 540m^5 + 2016m^4 - 816m^3 - 60m + 36, \\
a_{0,6}(m) &= 60m^7 + 180m^6 + 840m^5 - 280m^4 - 344m^3 + 36m^2 + 52m - 4.
\end{aligned}$$

Our purpose is to show that $\delta_0(m,k)$ and so $D(m,k)$ are positive for all $k \geq 4$ and $2 \leq m \leq \frac{2}{3}k$. Observe that $\delta_0(2,k) = 39092k^6 - 67668k^5 - 670046k^4 + 1782480k^3 - 1216850k^2 + 309000k - 25000$, and its greatest real zero is approximately equal to 3.5943 which means that $\delta_0 > 0$ for $m = 2$ and $k \geq 4$. Similarly, $\delta_0(3, k) = 435068k^6 - 1022364k^5 - 11599695k^4 + 37943730k^3 - 27453327k^2 + 7331148k - 644204$, has its greatest real zero approximately at 4.6413 so that $\delta_0 > 0$ for $m = 3$ and $k \geq 5 > \frac{3}{2}m > 4.5$.

We can focus on proving that $\delta_0(m, k) > 0$ for $m \geq 4$ and $k \geq \frac{3}{2}m$. To this end we introduce polynomials $\delta_j(m, k) = \delta_{j-1}(m, k + 1) - \delta_{j-1}(m, k)$, $j = 1, 2, 3, 4$, of the form $\delta_j(m, k) = \sum_{i=0}^{6-j} a_{ji}(m)k^i$ with coefficients

$$
\begin{aligned}
a_{1,0}(m) &= 76m^9 + 504m^8 + 759m^7 - 574m^6 - 1677m^5 - 248m^4 + 664m^3 + 48m^2 \\
&\quad - 124m + 32, \\
a_{1,1}(m) &= 216m^9 + 1292m^8 + 792m^7 - 4826m^6 - 4832m^5 + 3036m^4 + 782m^3 \\
&\quad - 1476m^2 + 876m - 144, \\
a_{1,2}(m) &= 480m^9 + 3732m^8 + 6486m^7 - 5538m^6 - 15126m^5 - 684m^4 + 6594m^3 \\
&\quad + 1368m^2 - 1788m + 264, \\
a_{1,3}(m) &= -256m^9 - 1760m^8 - 8700m^7 - 19896m^6 + 992m^5 + 25832m^4 \\
&\quad - 1716m^3 - 11448m^2 + 4132m - 248, \\
a_{1,4}(m) &= -240m^7 - 960m^6 + 9900m^5 + 5880m^4 - 9240m^3 + 540m^2 + 480m + 120, \\
a_{1,5}(m) &= 360m^7 + 1080m^6 + 5040m^5 - 1680m^4 - 2064m^3 + 216m^2 + 312m - 24, \\
a_{2,0}(m) &= 440m^9 + 3264m^8 - 1302m^7 - 30140m^6 - 4026m^5 + 32384m^4 \\
&\quad - 5644m^3 - 10800m^2 + 4012m - 32, \\
a_{2,1}(m) &= 192m^9 + 2184m^8 - 12288m^7 - 69204m^6 + 37524m^5 + 91248m^4 \\
&\quad - 39240m^3 - 28368m^2 + 12300m + 144, \\
a_{2,2}(m) &= -768m^9 - 5280m^8 - 23940m^7 - 54648m^6 + 112776m^5 + 95976m^4 \\
&\quad - 81228m^3 - 28944m^2 + 18396m - 264, \\
a_{2,3}(m) &= 2640m^7 + 6960m^6 + 90000m^5 + 6720m^4 - 57600m^3 + 4320m^2 \\
&\quad + 5040m + 240, \\
a_{2,4}(m) &= 1800m^7 + 5400m^6 + 25200m^5 - 8400m^4 - 10320m^3 + 1080m^2 \\
&\quad + 1560m - 120, \\
a_{3,0}(m) &= -576m^9 - 3096m^8 - 31788m^7 - 111492m^6 + 265500m^5 + 185544m^4 \\
&\quad - 188388m^3 - 51912m^2 + 37296m, \\
a_{3,1}(m) &= -1536m^9 - 10560m^8 - 32760m^7 - 66816m^6 + 596352m^5 + 178512m^4 \\
&\quad - 376536m^3 - 40608m^2 + 58152m - 288, \\
a_{3,2}(m) &= 18720m^7 + 53280m^6 + 421200m^5 - 30240m^4 - 234720m^3 \\
&\quad + 19440m^2 + 24480m,
\end{aligned}
$$

$$\begin{aligned}
a_{3,3}(m) &= 7200m^7 + 21600m^6 + 100800m^5 - 33600m^4 - 41280m^3 + 4320m^2 \\
&+ 6240m - 480, \\
a_{4,0}(m) &= -1536m^9 - 10560m^8 - 6840m^7 + 8064m^6 + 1118352m^5 + 114672m^4 \\
&- 652536m^3 - 16848m^2 + 88872m - 768, \\
a_{4,1}(m) &= 59040m^7 + 171360m^6 + 1144800m^5 - 161280m^4 - 593280m^3 \\
&+ 51840m^2 + 67680m - 1440, \\
a_{4,2}(m) &= 21600m^7 + 64800m^6 + 302400m^5 - 100800m^4 - 123840m^3 + 12960m^2 \\
&+ 18720m - 1440.
\end{aligned}$$

We first show that $\delta_4(m,k)$, which is quadratic function of argument k, is positive for all $m \geq 4$ and $k \geq \frac{3}{2}m$. Coefficient $a_{4,2}(m)$ has the greatest real zero approximately equal to 0.5121. Therefore $a_{4,2}(m) > 0$ for all $m \geq 4$. In order to prove positivity of $\delta_4(m,k)$ for all $m \geq 4$ and $k \geq \frac{3}{2}m$, it suffices to verify that $\delta_4(m,0) < 0 < \delta_4\left(m, \frac{3m}{2}\right)$. The first relation follows from the fact that $\delta_4(m,0) = a_{4,0}(m)$ has the greatest zero at 3.9567, i.e. it is negative for $m \geq 4$. The latter one is true because $\delta_4\left(m, \frac{3m}{2}\right) = 47064m^9 + 223800m^8 + 930600m^7 + 1498464m^6 + 597792m^5 - 746088m^4 - 532656m^3 + 81432m^2 + 86712m - 768$ has the greatest zero at 0.5127, and it positive for greater m. So we proved that $\delta_4(m,k) > 0$ for $m \geq 4$ and $k \geq \frac{3}{2}m$.

We show the same for δ_3. For fixed m, the smallest possible values of k are $\frac{3m}{2}$ if m is even and $\frac{3m+1}{2}$ if it is odd. Since

$$\begin{aligned}
\delta_3\left(m, \frac{3m}{2}\right) &= 21996m^{10} + 98604m^9 + 407844m^8 + 702288m^7 + 575676m^6 \\
&+ 19728m^5 - 314460m^4 - 195840m^3 + 35316m^2 + 36864m, \\
\delta_3\left(m, \frac{3m+1}{2}\right) &= 21996m^{10} + 122136m^9 + 511644m^8 + 1135908m^7 + 1190088m^6 \\
&+ 213324m^5 - 620904m^4 - 392868m^3 + 62532m^2 + 72300m - 204,
\end{aligned}$$

are ultimately positive, and have greatest zeros equal to 0.5174 and 0.5155, respectively. They are positive for all $m \geq 4$. Positivity for greater k follow by induction from the relations $\delta_3(m,k+1) = \delta_3(m,k) + \delta_4(m,k) > 0$, because both summands are greater than 0.

In the same manner we can prove the same conclusions for δ_j, $j = 2, 1,$ and 0. It suffices to check that the leading terms of $\delta_j\left(m, \frac{3m}{2}\right)$ and $\delta_j\left(m, \frac{3m+1}{2}\right)$ have positive coefficients, and respective maximal zeros are less than 4. Indeed, we have

$$\begin{aligned}
\delta_2\left(m, \frac{3m}{2}\right) &= \frac{14769}{2}m^{11} + \frac{49311}{2}m^{10} + 100916m^9 + 123099m^8 + 119073m^7 \\
&+ \frac{106319}{2}m^6 - \frac{54879}{2}m^5 - \frac{150395}{2}m^4 - 5995m^3 + 7056m^2 \\
&+ 4228m - 32, \\
\delta_2\left(m, \frac{3m+1}{2}\right) &= \frac{14769}{2}m^{11} + \frac{71307}{2}m^{10} + 144335m^9 + 303096m^8 + \frac{738789}{2}m^7 \\
&+ 220087m^6 - \frac{88749}{2}m^5 - \frac{341783}{2}m^4 - 70048m^3 + \frac{42165}{2}m^2 \\
&+ \frac{30977}{2}m - \frac{7}{2}, \\
\delta_1\left(m, \frac{3m}{2}\right) &= \frac{7479}{4}m^{12} + \frac{8505}{4}m^{11} + 12771m^{10} - \frac{52721}{4}m^9 + \frac{13347}{2}m^8 \\
&+ \frac{6129}{4}m^7 - \frac{20099}{2}m^6 - \frac{74703}{4}m^5 + 18556m^4 - 6410m^3 \\
&+ 1956m^2 - 340m + 32,
\end{aligned}$$

$$\begin{aligned}
\delta_1\left(m, \frac{3m+1}{2}\right) &= \frac{7479}{4}m^{12} + \frac{11637}{2}m^{11} + \frac{89397}{4}m^{10} + \frac{111575}{4}m^9 + \frac{114795}{4}m^8 \\
&+ \frac{87531}{4}m^7 - 2894m^6 - \frac{102723}{4}m^5 - \frac{28513}{4}m^4 + \frac{6011}{4}m^3 \\
&+ \frac{8571}{4}m^2 + 403m + \frac{7}{4}, \\
\delta_0\left(m, \frac{3m}{2}\right) &= \frac{5751}{16}m^{13} - \frac{14985}{16}m^{12} + \frac{17487}{16}m^{11} - \frac{38871}{4}m^{10} + \frac{55325}{4}m^9 \\
&- \frac{120585}{16}m^8 - \frac{31243}{4}m^7 - \frac{2429}{16}m^6 + \frac{263301}{16}m^5 - 16144m^4 \\
&+ \frac{29373}{4}m^3 - 1956m^2 + 340m - 32, \\
\delta_0\left(m, \frac{3m+1}{2}\right) &= \frac{5751}{16}m^{13} - \frac{27}{16}m^{12} + 1233m^{11} - \frac{80631}{16}m^{10} - \frac{16785}{16}m^9 \\
&- \frac{4269}{4}m^8 - \frac{62021}{8}m^7 - \frac{92313}{16}m^6 + \frac{58641}{8}m^5 - \frac{9333}{8}m^4 \\
&- \frac{47}{4}m^3 - \frac{231}{4}m^2 + \frac{279}{8}m - \frac{27}{16}
\end{aligned}$$

satisfy the first requirement, and their greatest zeros are approximately equal to 0.5468, 0.5291, 1.0630, 0.6453, 3.3136, and 2.2059, respectively. This completes the proof that $\delta_0(m,k)$ and so $D(m,k)$ are positive for every $m \geq 2$ and $k \geq \frac{3}{2}m$, and implies that each respective $d_{m,k}$ has a single real zero where it changes its sign from minus to plus.

In order to show that $d_{m,k} > 0$ on $(x_{f,1}(m,k), x_{f,2}(m,k))$, it is enough to check the condition at $x_{f,1}(m,k)$ only. We have

$$d_{m,k}(x_{f,1}(m,k)) = \frac{k^3 m\sqrt{m}(m+1)}{(k-2)^2} a(m,k),$$

where

$$\begin{aligned} a(m,k) &= \sqrt{m}[k(2k^2+k-19)+3m(k-1)+18] \\ &- [(k+1)(k-2)+3m(k-1)]\sqrt{4(k-1)(k-2)+m}. \end{aligned}$$

Since both terms are positive for $k \geq 4$ and $2 \leq m \leq \frac{2}{3}k$, we can verify positivity of the difference of their squares. This can be reduced to the inequality

$$4k(k-1)(k-2)^3(m-1)[(k+1)^2-6m] > 0$$

which is true under our restrictions. Accordingly, positivity of $d_{m,k}$ on interval $(x_{f,1}(m,k), x_{f,2}(m,k))$ implies negativity of $p_{m,k}$ and $h_{2,m,k}$ there.

Now we prove that $h_{2,m,k}$ is first negative and then positive in $(x_{f,2}(m,k),\infty)$ when $k \geq 3$ and $m \geq 2$ (assumption $k \geq \frac{3}{2}m$ is redundant in this case). We use Lemma 9(i) with $f = f_{2,m,k}$, $g = g_{2,m,k}$, $n = m$, and $x_f = x_{f,2}(m,k)$. Obviously, $f_{2,m,k}(x) \nearrow \infty$ as $x \nearrow \infty$. Condition $x_{f,2}(m,k) > m$ is first verified for $m = 2$. We have

$$x_{f,2}(2,k) = \frac{2k-3+\sqrt{2(k-1)(k-2)+1}}{(k-1)(k-2)} k > 2, \qquad k \geq 2,$$

iff

$$k\sqrt{2(k-1)(k-2)+1} > 2(k-1)(k-2)-(2k-3)k = -3k+4.$$

This is obvious, because the left- and right-hand sides of the inequality are positive and negative, respectively. We also check that numerator and denominator of

$$\frac{\partial}{\partial m}[x_{f,2}(m,k)-m] = \frac{(3k-4)\sqrt{m[4(k-1)(k-2)+m]}+2k(k-1)(k-2)+km}{2(k-1)(k-2)\sqrt{m[4(k-1)(k-2)+m]}}$$

are greater than 0, which implies that $x_{f,2}(m,k) > m$ for every $k \geq 3$ and $m \geq 2$. Further, we easily see that

$$g_{2,m,k}(x_{f,2}(m,k)) = \frac{mk^2}{(k-2)^2}\left[2k(k-1)+3m-4+3\sqrt{m[4(k-1)(k-2)+m]}\right] > 0.$$

Since

$$\frac{d}{dx}\frac{g_{2,m,k}(x)}{f_{2,m,k}(x)} = \frac{2km}{f_{2,m,k}^2(x)} a_{m,k}(x),$$

where

$$a_{m,k}(x) = -3(k-1)^2x^2 + 2k(k-1)(3m-k-1)x + k^2m(2k-3m),$$

it suffices to check that $a_{m,k}(x) < 0$ for $x > x_{f,2}(m,k)$. Note that the coefficient at the square term is negative, and $\Delta_a = 4k^2(k-1)^2[(k+1)^2 - 6m]$. If $m > \frac{(k+1)^2}{6}$, then $a_{m,k}(x) < 0$ for all $x \in \mathbb{R}$, as desired. For $m \leq \frac{(k+1)^2}{6}$, we prove that the greater zero of $a_{m,k}$ (unique if $\Delta_a = 0$), equal to $x_{2,a}(m,k) = \frac{k}{3(k-1)}\left(3m - k - 1 + \sqrt{(k+1)^2 - 6m}\right)$ is smaller than $x_{f,2}(m,k)$. We have

$$\begin{aligned} & 6(k-1)(k-2)[x_{f,2}(m,k) - x_{2,a}(m,k)] = 3\sqrt{m[4(k-1)(k-2)+m]} \\ + \quad & [2(k+1)(k-2)+3m] - 2(k-2)\sqrt{(k+1)^2 - 6m}. \end{aligned}$$

It suffices to show that the difference of the squares of two expressions in the second line is positive. This relation can be simplified to $9m[4(k-1)(k-2)+m] > 0$ which is evidently true. This completes verification of assumptions in Lemma 9(i) and sign analysis of $h_{2,m,k}$ in $(x_{f,2}(m,k), \infty)$.

Summing up, for all $m \geq 2$ and $k \geq \frac{3}{2}m$, function $h_{2,m,k}(x)$, $x > 0$, is first positive, then negative and finally positive again. If follows that each respective $\psi_{m,k}(u)$, $0 < u < 1$, is first convex, then concave, and finally convex. Since it also vanishes at 0 and 1, and is positive, and has a single maximum in between, it is first convex increasing, then concave increasing and ultimately decreasing. By Lemma 7, $\Psi_{m,k}$ is also increasing-decreasing, and its unique local maximum is global as well.

(iv) If $k \geq 3$ and $m > \frac{2}{3}k$, then $\lim_{u \searrow 0} \Psi_{m,k}(u) = \lim_{u \nearrow 1} \Psi_{m,k}(u) = 0$, and so the supremum of $\Psi_{m,k}$ is attained at some inner point u_0 of $(0,1)$. □

Proof of Remark 10. Using (5.3.9)–(5.3.11) for $m, k \geq 3$, and expanding the exponential function into the Taylor series, we obtain

$$\begin{aligned}
h_{2,m,k}(x) &= k^2m(m-1) + \big[k^2m(m-1) - (2k-3)km\big]x \\
&+ \big[k^2m(m-1) - (2k-3)km^2 + (k-1)(k-2)m(m-1) - 2k^2m(2m-1)\big]\frac{x^m}{m!} \\
&+ \big[k^2m(m-1) - (2k-3)km(m+1) + (k-1)(k-2)m(m+1) \\
&+ 2k(4k-3)m(m+1)\big]\frac{x^{m+1}}{(m+1)!} \\
&+ \big[-(2k-3)km(m+2) + (k-1)(k-2)(m+1)(m+2) \\
&- k^2m(m-1) + 2(k-1)(2k-1)(m+1)(m+2)\big]\frac{x^{m+2}}{(m+2)!} \\
&+ \sum_{i\in\mathbb{N}\setminus\{1,m,m+1,m+2\}} \big[k^2m(m-1) - (2k-3)kmi + (k-1)(k-2)i(i-1)\big]\frac{x^i}{i!} \\
&= \sum_{i=0}^{\infty} a_{m,k}(i)\frac{x^i}{i!}.
\end{aligned}$$

Clearly, $a_{m,k}(0) > 0$ and $a_{m,k}(1) = [(m-3)k+3]km > 0$. Moreover

$$\begin{aligned}
a_{m,k}(m) &= -m\big[(k^2-2)m + 3k(km-1) + 2\big] < 0, \\
a_{m,k}(m+1) &= 2m\big[(4k-3)km + 3k(k-1) + m + 1\big] > 0, \\
a_{m,k}(m+2) &= -k(2m+1)[m(2k-3) + 6(k-1)] < 0.
\end{aligned}$$

Function

$$a_{m,k}(i) = k^2m(m-1) - (2k-3)kmi + (k-1)(k-2)i(i-1)$$

is decreasing-increasing with the minimum at

$$b_{m,k} = \frac{(2k-3)km + (k-1)(k-2)}{2(k-1)(k-2)}$$

One can easily check that $b_{m,k} > m$. This implies that the sign sequence for $(a_{m,k})_{i=0}^{m+2}$ is $+-+-$. We also note that

$$\begin{aligned}
-\frac{4(k-1)(k-2)}{mk}a_{m,k}\left(b_{m,k} \mp \frac{1}{2}\right) &= 8k^3 - 30k^2 + km + 34k - 12 \\
&= 8(k-2)^3 + 18(k-2)^2 + 10(k-2) + km > 0
\end{aligned}$$

which implies that $a_{m,k}$ is negative for at least one integer i at the neighborhood of $b_{m,k}$ for every $k, m \geq 3$. Furthermore, $a_{m,k}(m+3) > 0$ iff

$$m > \mu(k) = \frac{1}{2}\left(k^2 + 3k - 5 + \sqrt{k^4 + 6k^3 - 13k^2 + 6k + 1}\right),$$

and $b_{m,k} > m+3$ iff

$$m > \nu(k) = 5\frac{(k-1)(k-2)}{3k-4}.$$

Since $\mu(k) > \nu(k)$ for all $k \geq 3$, relation $a_{m,k}(m+3) > 0$ implies $b_{m,k} > m+3$. Therefore $(a_{m,k})_{i=m+3}^{\infty}$ has the sign sequence $+-+$ if $m > \mu(k)$, and $-+$ when $m \leq \mu(k)$. Since $\mu(k) > \frac{2}{3}k$, we conclude from Lemma 4 that for $k \geq 3$, under assumptions $\frac{2}{3}k < m \leq \mu(k)$, and $m > \mu(k)$, the maximal possible sign changes of $h_{2,m,k}$ and $\psi''_{m,k}$ are $+-+-+$ and $+-+-+-+$, respectively. It follows that $\psi_{m,k}$ may possibly have 2 or 3 intervals of concave increase then, and $\Psi_{m,k}$ may have at most 1 local maximum in each of them. □

Proposition 15. *Under assumptions of Proposition 14, with the exception of case $m = k = 1$, bound*

$$\frac{\mathbb{V}ar(R_{m+1,k} - R_{m,k})}{\mathbb{V}ar\, X_1} \geq 0$$

is the best possible. In particular,
(i) if $m \geq 2$, and $k \geq 1$, then

$$\lim_{u \searrow 0} \Psi_{m,k}(u) = 0,$$

and the zero bound is attained under conditions of Theorem 6(ii),
(ii) if $m \geq 1$, and $k \geq 2$, then

$$\lim_{u \nearrow 1} \Psi_{m,k}(u) = 0,$$

and the zero bound is attained under conditions of Theorem 6(iii).

Proof. In view of Theorem 7, it suffices to notice that

$$\begin{aligned} \lim_{u \searrow 0} \Psi_{m,k}(u) &= 0, \qquad k \geq 2, \\ \lim_{u \nearrow 1} \Psi_{m,k}(u) &= 0, \qquad m \geq 2. \end{aligned}$$

□

Remark 11. *Calculating the sharp lower bounds for the case $k = m = 1$ is an open problem. We have*

$$\inf_{0<u<1} \Psi_{1,1}(u) = \Psi_{1,1}(0.37977) = 0.88514$$

which is the lowest possible value attained by the sequences of continuous distributions tending weakly to some two-point ones , as in Theorem 6(i). However, when we consider the family of Weibull baseline distribution functions $F_\alpha(x) = 1 - \exp(-x^\alpha)$, $x > 0$, with shape parameter $\alpha > 0$ (the scale parameter does not matter here), then we obtain

$$\begin{aligned}
&\frac{\mathbb{V}ar_\alpha(R_{2,1} - R_{1,1})}{\mathbb{V}ar_\alpha X_1} = V(\alpha) \\
= \;& \frac{\Gamma(1+1/\alpha) + \Gamma(2+2/\alpha) - 2\Gamma(1+1/\alpha)\frac{\Gamma(2+2/\alpha)}{\Gamma(2+1/\alpha)} - [\Gamma(2+1/\alpha) - \Gamma(1+1/\alpha)]^2}{\Gamma(1+2/\alpha) - \Gamma^2(1+1/\alpha)} \\
\geq \;& V(4.88090) = 0.57492
\end{aligned}$$

(cf Arnold et al, 1998, p. 55).

Table 5.1: Upper bounds on variances of kth record spacings $\frac{\mathbb{V}ar(R_{m+1,k} - R_{m,k})}{\mathbb{V}ar\, X_1}$ for $k = 2, 3, 4$ and $m = 2, \ldots, 8$.

k	2		3		4	
m	$u_0(m,k)$	$\Psi_{m,k}(u_0)$	$u_0(m,k)$	$\Psi_{m,k}(u_0)$	$u_0(m,k)$	$\Psi_{m,k}(u_0)$
2	0.86934	1.06896	0.44816	0.79186	0.26639	0.88481
3	0.95642	1.72309	0.75606	0.81215	0.55201	0.69942
4	0.98432	3.01160	0.86434	0.99074	0.70792	0.70797
5	0.99419	5.45217	0.92156	1.29382	0.80154	0.78566
6	0.99782	10.0581	0.95377	1.75316	0.86280	0.91637
7	0.99918	18.7747	0.97243	2.42904	0.90421	1.10242
8	0.99969	35.3359	0.98344	3.41419	0.93267	1.35393

Table 5.1 depicts upper bounds on ratios $\frac{\mathbb{V}ar(R_{m+1,k} - R_{m,k})}{\mathbb{V}ar\, X_1}$ equal to $\Psi_{m,k}(u_0)$ for $k = 2, 3, 4$ and $m = 2, \ldots, 8$ and arguments u_0 for which respective functions $\Psi_{m,k}$ attain their maxima. We can observe that when m increases then both arguments u_0 and values $\Psi_{m,k}(u_0)$ increase for all $k \in \{2, 3, 4\}$ except $k = 4$ and $m = 2, 3$. One can expect that for greater k, the bounds first decrease and then increase in m. It is also seen that when k increases then both arguments u_0 and $\Psi_{m,k}(u_0)$ decreases for all $m \in \{2, \ldots, 8\}$ except $k = 3, 4$ and $m = 2$. Greater values of bounds occur for small k as m increases.

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Summary

This dissertation is devoted to determination of sharp bounds on the expectations and variances of linear combinations of order statistics and kth records based on independent and identically distributed random variables. Its novelty consists in the following. The bounds on the expectations are expressed in the scale units being the Gini mean difference of the population. Bounds on variances of single order statistics and records are extended to the case of proper linear combinations. The main idea of our reasoning consists in integral representation of the expectations, variances and covariances of order and record statistics so that the integrand is the composition of some (usually complicated) function with the baseline distribution function. Below we present our general results with some exemplary special cases.

Bounds on the expectations of L-statistics

Suppose that $X_1, \ldots, X_n$ are non-degenerate i.i.d. random variables with a finite mean $\mu = \mathbb{E}X_1$. Let $X_{1:n} \leq \ldots \leq X_{n:n}$ stand for the respective order statistics. Firstly we focus on determining of sharp lower and upper bounds on the expectations of properly centered L-statistics $\mathbb{E}\sum_{i=1}^{n} c_i(X_{i:n} - \mu)$, with arbitrary $c_1, \ldots, c_n \in \mathbb{R}$ and their special cases, expressed in terms of the Gini mean difference scale units $\Delta = \mathbb{E}|X_1 - X_2|$. Given $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$ with the arithmetic mean $\bar{c} = \frac{1}{n}\sum_{i=1}^{n} c_i$, we define function

$$\Xi_{\mathbf{c}}(u) = \sum_{i=0}^{n-2} \frac{n(n-1)}{2(i+1)(n-i-1)} \left[\sum_{k=1}^{i+1} (\bar{c} - c_k)\right] \binom{n-2}{i} u^i (1-u)^{n-2-i}$$

on the interval $[0, 1]$.

Theorem (see Theorem 2) *Under the above assumptions and notation, the following bounds are optimal*

$$\min_{0 \leq u \leq 1} \Xi_{\mathbf{c}}(u) \leq \mathbb{E}\, \frac{\sum_{i=1}^{n} c_i(X_{i:n} - \mu)}{\Delta} \leq \max_{0 \leq u \leq 1} \Xi_{\mathbf{c}}(u).$$

If $0 < u_0 < 1$ *is the argument of the maximum (minimum, respectively), then the upper (lower) bound is attained iff the parent distribution function has the form*

$$F(x) = \begin{cases} 0, & x < a, \\ u_0, & a \le x < b, \\ 1, & x \ge b, \end{cases}$$

for arbitrary $a < b$.

If the maximum (minimum) amounts either to $\Xi_{\mathbf{c}}(0)$ *or to* $\Xi_{\mathbf{c}}(1)$*, then the upper (lower, respectively) bound is attained in the limit by the two-point distributions such that the probabilities of the smaller point tend to* 0 *and* 1*, respectively.*

If $\Xi_{\mathbf{c}}$ has multiple extremes which happens very rarely other discrete distributions may attain the bounds as well. We do not discuss them here. For brevity of presentation, further on we use the following convention. Writing either the lower or the upper bound as $\Xi_{\mathbf{c}}(u_0)$, say, for some $0 < u_0 < 1$, we mean that the bound is attained by the two-point distribution described in the Theorem. If the bound is either $\Xi_{\mathbf{c}}(0)$ or $\Xi_{\mathbf{c}}(0)$, the bounds are attained in the limit by the two-point distributions with probability of the smaller point tending to 0 and 1, respectively.

For single order statistics $X_{r:n}$, $1 \le r \le n$, $\Xi_{\mathbf{c}}$ simplifies to

$$\Xi_{r:n}(u) = \sum_{i=0}^{r-2} \frac{n-1}{2(n-i-1)} \binom{n-2}{i} u^i (1-u)^{n-2-i} - \sum_{i=r-1}^{n-2} \frac{n-1}{2(i+1)} \binom{n-2}{i} u^i (1-u)^{n-2-i}.$$

Proposition (see Proposition 1) *(i) For the extreme order statistics, we have*

$$\Xi_{1:n}(0) = -\frac{n-1}{2} \le \mathbb{E}\frac{X_{1:n}-\mu}{\Delta} \le \Xi_{1:n}(1) = -\frac{1}{2},$$
$$\Xi_{n:n}(0) = \frac{1}{2} \le \mathbb{E}\frac{X_{n:n}-\mu}{\Delta} \le \Xi_{n:n}(1) = \frac{n-1}{2}.$$

(ii) For the second extremes, the derivatives $\Xi'_{r:n}(u)$, $r = 2, n-1$, *have unique zeros* $v_1(2)$ *and* $u_1(n-1) = 1 - v_1(2)$*, respectively, and*

$$\Xi_{2:n}(v_1(2)) \le \mathbb{E}\frac{X_{2:n}-\mu}{\Delta} \le \Xi_{2:n}(0) = \frac{1}{2},$$
$$\Xi_{n-1:n}(1) = -\frac{1}{2} \le \mathbb{E}\frac{X_{n-1:n}-\mu}{\Delta} \le \Xi_{n-1:n}(u_1(n-1)).$$

(iii) For $3 \le r \le n-2$, $\Xi'_{r:n}(u)$ *has two zeros* $u_1(r) < v_1(r)$ *in* $(0,1)$*, and*

$$\Xi_{r:n}(v_1(r)) \le \mathbb{E}\frac{X_{r:n}-\mu}{\Delta} \le \Xi_{r:n}(u_1(r)),$$

In a similar way, we evaluate the expectations of differences of two order statistics, trimmed and Winsorized means, and their differences, and mean deviation from the median.

Bounds on the variances of L-statistics

We consider i.i.d. random variables $X_1, \ldots, X_n$ with a positive and finite variance. Define

$$\begin{aligned}\Phi_{\mathbf{c}}(u,v) &= \left[\sum_{i=0}^{n-1} \frac{n}{i+1}\left(\sum_{k=1}^{i+1} c_k\right)\binom{n-1}{i} u^i (1-u)^{n-1-i}\right] \\ &\times \left[\sum_{j=0}^{n-1} \frac{n}{n-j}\left(\sum_{m=j+1}^{n} c_m\right)\binom{n-1}{j} v^j (1-v)^{n-1-j}\right] \\ &- \sum_{i=0}^{n-2}\sum_{j=i}^{n-2} \frac{n(n-1)}{(i+1)(n-1-j)}\left(\sum_{k=1}^{i+1} c_k\right)\left(\sum_{m=j+2}^{n} c_m\right) \\ &\times \frac{n!}{i!(j-i)!(n-j)!} u^i (v-u)^{j-i} (1-v)^{n-2-j}.\end{aligned}$$

for $0 < u \leq v < 1$ and set $\Psi_{\mathbf{c}}(u) = \Phi_{\mathbf{c}}(u,u)$.

Theorem (see Theorem 3) *Under the above assumptions and notation, we have*

$$\frac{\mathbb{V}ar(\sum_{i=1}^n c_i X_{i:n})}{\mathbb{V}ar\, X_1} \leq \sup_{0<u\leq v<1} \Phi_{\mathbf{c}}(u,v).$$

Moreover, if

$$\sup_{0<u\leq v<1} \Phi_{\mathbf{c}}(u,v) = \sup_{0<u<1} \Psi_{\mathbf{c}}(u),$$

then the bound is sharp.

We also show that the lower bounds on the ratio $\frac{\mathbb{V}ar(\sum_{i=1}^n c_i X_{i:n})}{\mathbb{V}ar\, X_1}$ trivially vanish iff $c_1 c_n = 0$. In the case of spacings $X_{i+1:n} - X_{i:n}$, $1 \leq i < n < \infty$, function $\Psi_{\mathbf{c}}$ takes on the form

$$\Psi_{i:n}(u) = \binom{n}{i} u^{i-1}(1-u)^{n-i-1}\left[1 - \binom{n}{i} u^i (1-u)^{n-i}\right].$$

Proposition (see Propositions 9 and 10) *The following bounds are sharp.*
(i) If integer $n \geq 3$*, then*

$$\begin{aligned} 0 = \Psi_{1:n}(1) \leq & \tfrac{\mathbb{V}ar(X_{2:n}-X_{1:n})}{\mathbb{V}ar\, X_1} & \leq \Psi_{1:n}(0) = n, \\ 0 = \Psi_{n-1:n}(0) \leq & \tfrac{\mathbb{V}ar(X_{n:n}-X_{n-1:n})}{\mathbb{V}ar\, X_1} & \leq \Psi_{n-1:n}(1) = n. \end{aligned}$$

(ii) If integer $n \geq 4$ *and* $2 \leq i \leq n-2$*, then* $\Psi'_{i:n}$ *has either one or three zeros, and*

$$0 = \Psi_{i:n}(0) = \Psi_{i:n}(1) \leq \frac{Var(X_{i+1:n} - X_{i:n})}{\mathbb{V}ar\, X_1} \leq \Psi_{i:n}(u_0),$$

where u_0 *is either the single zero of the derivative, or the first or third zero of* $\Psi'_{i:n}$ *otherwise. The argument is chosen then so that it provides the greater value of* $\Psi_{i:n}$*.*

Case $n = 2$ with $i = 1$ for which the lower bound is strictly positive is not discussed in this summary.

Bounds on the expectations of linear combinations of records

Let $X_1, X_2, \ldots$ be i.i.d. random variables with finite mean μ. Let $R_{1,k}, R_{2,k}, \ldots$ denote respective kth record values. We assume that $\mathbb{E}R_{n,k} < \infty$ for some fixed n and k. Below we describe sharp lower and upper bounds for expectations of arbitrary linear combinations of kth records $\mathbb{E}\left[\sum_{i=1}^n c_i(R_{i,k} - \mu)\right]$, centered about the population mean, and expressed in the Gini mean difference units $\Delta = \mathbb{E}|X_1 - X_2|$. We use the following notation

$$\begin{aligned} \Xi_{n,k}(u) &= \frac{1}{2u}\left[(1-u)^{k-1}\sum_{i=0}^{n-1}\frac{\left[-k\ln(1-u)\right]^i}{i!} - 1\right], \\ \Xi_{\mathbf{c},k}(u) &= \sum_{i=1}^{n} c_i \Xi_{i,k}(u) = \frac{1}{2u}\left[(1-u)^{k-1}\sum_{i=0}^{n-1}\left(\sum_{j=i+1}^{n} c_j\right)\frac{[-k\ln(1-u)]^i}{i!} - \sum_{j=1}^{n} c_j\right], \end{aligned}$$

where $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$.

Theorem (see Theorem 5) *Under the above assumptions and notation, the following bounds*

$$\inf_{0<u<1} \Xi_{\underline{c},k}(u) \leq \frac{\mathbb{E}\left[\sum_{i=1}^n c_i(R_{i,k} - \mu)\right]}{\Delta} \leq \sup_{0<u<1} \Xi_{\underline{c},k}(u)$$

are sharp.

The attainability conditions are similar to these in the expectation problems. The only differences are that exact values at the end-points 0 and 1 are replaced by respective limits, and discrete two-points distributions are replaced by their continuous approximations. Accordingly, all bounds for the combinations of records are attained in the limit. For single records $R_{n,k}$, we have $\Xi_{\mathbf{c},k} = \Xi_{n,k}$.

Proposition (see Proposition 12) *For various natural $n \geq 2$ and $k \geq 1$, we have the following sharp bounds.*
(i) For $n \geq 2$ and $k = 1$, yields

$$\frac{1}{2} = \Xi_{n,1}(0+) \leq \frac{\mathbb{E}(R_{n,1} - \mu)}{\Delta} \leq \Xi_{n,1}(1-) = \infty.$$

(ii) If $n = k = 2$, then

$$-\frac{1}{2} = \Xi_{2,2}(1-) \leq \frac{\mathbb{E}(R_{2,2} - \mu)}{\Delta} \leq \Xi_{2,2}(0+) = \frac{1}{2}.$$

(iii) For $n \geq 3$ and $k = 2$

$$-\frac{1}{2} = \lim_{u \nearrow 1^-} \Xi_{n,2}(1-) \leq \frac{\mathbb{E}(R_{n,2} - \mu)}{\Delta} \leq \Xi_{n,2}(u_1) > \frac{1}{2},$$

where u_1 is the unique zero of $\Xi'_{n,2}$ in $(0, 1)$.
(iv) For $n = 2$ and $k \geq 3$

$$-\frac{1}{2} > \Xi_{2,k}(u_1) \leq \frac{\mathbb{E}(R_{2,k} - \mu)}{\Delta} \leq \Xi_{2,k}(0+) = \frac{1}{2},$$

where $u_1 \in (0, 1)$ is the unique solution zero of $\Xi'_{2,k}$ in $(0, 1)$.
(v) For $k \geq 3$ and $n \geq 3$, we have

$$-\frac{1}{2} > \Xi_{n,k}(u_2) \leq \frac{\mathbb{E}(R_{n,k} - \mu)}{\Delta} \leq \Xi_{n,k}(u_1) > \frac{1}{2},$$

with $0 < u_1 < u_2 < 1$ being the only two solutions to equation $\Xi'_{n,k}(u) = 0$.

Similarly, we evaluated the expectations of differences of kth record values $\mathbb{E}(R_{n,k} - R_{m,k})$, $1 \leq m < n$.

Variances of linear combinations of records

Let $X_1, X_2, \dots$ be i.i.d. random variables with finite second moment. Assume that $\mathbb{E}R_{n,k}^2 < \infty$. For given positive integers n and k, and for a fixed non-zero vector $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{R}^n$, we define functions

$$\begin{aligned}\Phi_{\mathbf{c},k}(u,v) &= \frac{(1-v)^{k-1}}{u}\left\{\left[\sum_{j=1}^{n} c_j - (1-u)^k \sum_{i=0}^{n-1}\left(\sum_{j=i+1}^{n} c_j\right)\frac{[-k\ln(1-u)]^i}{i!}\right]\right.\\ &\times \sum_{i=0}^{n-1}\left(\sum_{j=i+1}^{n} c_j\right)\frac{[-k\ln(1-v)]^i}{i!}\\ &- \left.\sum_{1\le i<j\le n} c_i c_j \sum_{p=0}^{j-i-1}\sum_{q=0}^{p}\frac{(-1)^q[-k\ln(1-u)]^{i+q}[-k\ln(1-v)]^{p-q}}{(i-1)!q!(p-q)!(p+i)}\right\}\end{aligned}$$

acting on the triangle $0 < u \le v < 1$, and $\Psi_{\mathbf{c},k}(u) = \Phi_{\mathbf{c},k}(u,u)$, $0 < u < 1$.

Theorem (see Theorem 6) *Under the above conditions and notation, we have*

$$\frac{\mathbb{V}ar(\sum_{i=1}^{n} c_i R_{i,k})}{\mathbb{V}ar\, X_1} \le \sup_{0<u\le v<1} \Phi_{\mathbf{c},k}(u,v).$$

Moreover, if

$$\sup_{0<u\le v<1} \Phi_{\mathbf{c},k}(u,v) = \sup_{0<u<1} \Psi_{\mathbf{c},k}(u),$$

then the bound is sharp.

We also proved that the lower bounds on the variance ratio vanish except for the case $k=1$ with $c_1 \neq 0$. For the kth record spacings $R_{m+1,k} - R_{m,k}$, function $\Psi_{\mathbf{c},k}$ has a simpler form

$$\Psi_{m,k}(u) = \frac{[-k\ln(1-u)]^m(1-u)^{k-1}}{um!}\left[1 - \frac{(1-u)^k[-k\ln(1-u)]^m}{m!}\right].$$

Proposition (see Proposition 14) *The following bounds are tight.*
(i) If $k=1$ and $m \ge 1$, then

$$\frac{\mathbb{V}ar(R_{m+1,1} - R_{m,1})}{\mathbb{V}ar\, X_1} \le \Psi_{m,1}(1-) = +\infty.$$

(ii) If $m=1$ and $k \ge 2$, then

$$\frac{\mathbb{V}ar(R_{2,k} - R_{1,k})}{\mathbb{V}ar\, X_1} \le \Psi_{1,k}(0+) = k.$$

(iii) If either $k = 2 \leq m$ *or* $k \geq 3$ *with* $2 \leq m \leq \frac{2}{3}k$*, then*

$$\frac{\mathbb{V}ar(R_{m+1,k} - R_{m,k})}{\mathbb{V}ar\, X_1} \leq \Psi_{m,k}(u_0),$$

where $0 < u_0 < 1$ *is the unique solution to* $\Psi'_{m,k}(u) = 0$*.*
(iv) If finally $k \geq 3$ *with* $m > \frac{2}{3}k$*, then*

$$\frac{\mathbb{V}ar(R_{m+1,k} - R_{m,k})}{\mathbb{V}ar\, X_1} \leq \Psi_{m,k}(u_0),$$

where $0 < u_0 < 1$ *is the global maximum point of* $\Psi_{m,k}$ *over* $(0, 1)$*.*

In the last case, we can formally prove that $\Psi_{m,k}$ has three local maxima at most.

Streszczenie

Dysertacja ta poświęcona jest wyznaczeniu optymalnych oszacowań wartości oczekiwanych i wariancji kombinacji liniowych statystyk pozycyjnych oraz k-tych rekordów skonstruowanych na bazie niezależnych zmiennych losowych o tym samym rozkładzie. Jest ona nowatorska w dwóch aspektach. Oszacowania wartości oczekiwanych zostały wyrażone w jednostkach średniej różnicy Giniego populacji, a oszacowania wariancji pojedynczych statystyk pozycyjnych i rekordów zostały uogólnione na przypadek nietrywialnych liniowych ich kombinacji. Zasadnicza idea naszego rozumowania polega na przedstawieniu wartości oczekiwanych, wariancji i kowariancji statystyk pozycyjnych i rekordowych w postaci całkowej w taki sposób, aby wyrażenie podcałkowe stanowiło złożenie pewnej funkcji (zwykle opisanej bardzo skomplikownym wzorem) z dystrybuantą rozważanych zmiennych losowych. Poniżej prezentujemy nasze ogólne rezultaty wraz z przykładowymi przypadkami szczególnymi.

Oszacowania wartości oczekiwanych L-statystyk

Przypuśćmy, że $X_1, \ldots, X_n$ są niezdegenerownymi, niezależnymi zmiennymi losowymi o tym samym rozkładzie ze skończoną średnią $\mu = \mathbb{E}X_1$. Niech $X_{1:n} \leq \ldots \leq X_{n:n}$ oznaczają ich statystyki pozycyjne. Najpierw koncentrujemy się na wyznaczeniu optymalnych oszacowań górnych i dolnych wartości oczekiwanych odpowiednio scentrowanych L-statystyk $\mathbb{E}\sum_{i=1}^{n} c_i(X_{i:n} - \mu)$, dla dowolnie ustalonych $c_1, \ldots, c_n \in \mathbb{R}$, oraz ich specjalnych przypadków. Są one wyrażone w jednostkach średniej różnicy Giniego $\Delta = \mathbb{E}|X_1 - X_2|$. Dla $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$ ze średnią arytmetyczną $\bar{c} = \frac{1}{n}\sum_{i=1}^{n} c_i$ definiujemy funkcję

$$\Xi_{\mathbf{c}}(u) = \sum_{i=0}^{n-2} \frac{n(n-1)}{2(i+1)(n-i-1)} \left[\sum_{k=1}^{i+1} (\bar{c} - c_k)\right] \binom{n-2}{i} u^i (1-u)^{n-2-i}$$

określoną na przedziale $[0, 1]$.

Twierdzenie (patrz Twierdzenie 2) *Przy powyższych założeniach i oznaczeniach, następujące*

oszacowania są optymalne

$$\min_{0\le u\le 1} \Xi_{\mathbf{c}}(u) \le \mathbb{E}\, \frac{\sum_{i=1}^{n} c_i(X_{i:n}-\mu)}{\Delta} \le \max_{0\le u\le 1} \Xi_{\mathbf{c}}(u).$$

Jeśli $0 < u_0 < 1$ *jest argumentem maksimum (odpowiednio minimum), to górne (dolne) oszacowanie jest osiągane wtedy i tylko wtedy, gdy dystrybuanta rozważanych zmiennych losowych ma postać*

$$F(x) = \begin{cases} 0, & x < a, \\ u_0, & a \le x < b, \\ 1, & x \ge b, \end{cases}$$

dla dowolnie ustalonych $a < b$.

Jeśli maksimum (minimum) wynosi albo $\Xi_{\mathbf{c}}(0)$ *albo* $\Xi_{\mathbf{c}}(1)$, *to górne (odpowiednio dolne) oszacowanie jest osiągane w granicy przez rozkłady dwu–punktowe takie, że prawdopodobieństwa mniejszego punktu zbiegają odpowiednio do* 0 *i* 1.

Jeżeli $\Xi_{\mathbf{c}}$ ma wielokrotne ekstrema, co zdarza się bardzo rzadko, to również inne rozkłady dyskretne osiągają te oszacowania. Nie omawiamy ich w tym streszczeniu. W celu skrócenia naszej prezentacji, będziemy też używać następującej konwencji. Pisząc, że dolne albo górne oszacowanie wynosi $\Xi_{\mathbf{c}}(u_0)$ dla pewnego $0 < u_0 < 1$, przyjmujemy milcząco, że oszacowanie to jest osiągane przez rozkłady dwu-punktowe opisane w powyższym twierdzeniu. Jeżeli oszacowanie jest równe $\Xi_{\mathbf{c}}(0)$ albo $\Xi_{\mathbf{c}}(1)$, to jest ono osiągane w granicy przez przez rozkłady dwu-punktowe o prawdopodobieństwie mniejszego punktu zbiegającym odpowiednio do 0 i 1.

Dla pojedynczej statystyki pozycyjnej $X_{r:n}$, $1 \le r \le n$, funkcja $\Xi_{\mathbf{c}}$ upraszcza się do

$$\Xi_{r:n}(u) = \sum_{i=0}^{r-2} \frac{n-1}{2(n-i-1)} \binom{n-2}{i} u^i (1-u)^{n-2-i} - \sum_{i=r-1}^{n-2} \frac{n-1}{2(i+1)} \binom{n-2}{i} u^i (1-u)^{n-2-i}.$$

Stwierdzenie (patrz Stwierdzenie 1) *(i) Dla skrajnych statystyk pozycyjnych mamy*

$$\begin{aligned} \Xi_{1:n}(0) = -\frac{n-1}{2} &\le \mathbb{E}\frac{X_{1:n}-\mu}{\Delta} \le \Xi_{1:n}(1) = -\frac{1}{2}, \\ \Xi_{n:n}(0) = \frac{1}{2} &\le \mathbb{E}\frac{X_{n:n}-\mu}{\Delta} \le \Xi_{n:n}(1) = \frac{n-1}{2}. \end{aligned}$$

(ii) Dla sąsiadujących ze skrajnymi statystyk pozycyjnych, pochodne $\Xi'_{r:n}(u)$, $r = 2, n-1$, *mają jedyne miejsca zerowe, odpowiednio* $v_1(2)$ *i* $u_1(n-1) = 1 - v_1(2)$, *i zachodzą nierówności*

$$\begin{aligned} \Xi_{2:n}(v_1(2)) &\le \mathbb{E}\frac{X_{2:n}-\mu}{\Delta} \le \Xi_{2:n}(0) = \frac{1}{2}, \\ \Xi_{n-1:n}(1) = -\frac{1}{2} &\le \mathbb{E}\frac{X_{n-1:n}-\mu}{\Delta} \le \Xi_{n-1:n}(u_1(n-1)). \end{aligned}$$

(iii) Dla $3 \leq r \leq n-2$, $\Xi'_{r:n}(u)$ ma dwa miejsca zerowe $u_1(r) < v_1(r)$ w $(0,1)$, i wówczas

$$\Xi_{r:n}(v_1(r)) \leq \mathbb{E}\frac{X_{r:n}-\mu}{\Delta} \leq \Xi_{r:n}(u_1(r)).$$

W podobny sposób szacujemy wartości oczekiwane różnic dwu statystyk pozycyjnych, uciętych i Winsoryzowanych średnich, ich różnic oraz średniego absolutnego odchylenia od mediany.

Oszacowania wariancji kombinacji liniowych statystyk pozycyjnych

Rozważamy niezależne zmienne losowe $X_1, \ldots, X_n$ o tym samym rozkładzie z niezerową i skończoną wariancją. Definiujemy

$$\begin{aligned}
\Phi_{\mathbf{c}}(u,v) &= \left[\sum_{i=0}^{n-1} \frac{n}{i+1}\left(\sum_{k=1}^{i+1} c_k\right)\binom{n-1}{i} u^i (1-u)^{n-1-i}\right] \\
&\times \left[\sum_{j=0}^{n-1} \frac{n}{n-j}\left(\sum_{m=j+1}^{n} c_m\right)\binom{n-1}{j} v^j (1-v)^{n-1-j}\right] \\
&- \sum_{i=0}^{n-2}\sum_{j=i}^{n-2} \frac{n(n-1)}{(i+1)(n-1-j)}\left(\sum_{k=1}^{i+1} c_k\right)\left(\sum_{m=j+2}^{n} c_m\right) \\
&\times \frac{n!}{i!(j-i)!(n-j)!} u^i (v-u)^{j-i} (1-v)^{n-2-j}.
\end{aligned}$$

dla $0 < u \leq v < 1$. Niech $\Psi_{\mathbf{c}}(u) = \Phi_{\mathbf{c}}(u,u)$.

Twierdzenie (patrz Twierdzenie 3) *Przy powyższych założeniach i oznaczeniach, zachodzi*

$$\frac{\mathbb{V}ar(\sum_{i=1}^{n} c_i X_{i:n})}{\mathbb{V}ar\, X_1} \leq \sup_{0<u\leq v<1} \Phi_{\mathbf{c}}(u,v).$$

Ponadto, jeżeli

$$\sup_{0<u\leq v<1} \Phi_{\mathbf{c}}(u,v) = \sup_{0<u<1} \Psi_{\mathbf{c}}(u),$$

to powyższe oszacowanie jest optymalne.

Pokazujemy także, że dolne oszacowanie ilorazu $\frac{\mathbb{V}ar(\sum_{i=1}^{n} c_i X_{i:n})}{\mathbb{V}ar\, X_1}$ jest równe 0 wtedy i tylko wtedy, gdy $c_1 c_n = 0$. W przypadku spacji $S_{i:n} = X_{i+1:n} - X_{i:n}$, $1 \leq i < n < \infty$, funkcja $\Psi_{\mathbf{c}}$ ma postać

$$\Psi_{i:n}(u) = \binom{n}{i} u^{i-1}(1-u)^{n-i-1}\left[1 - \binom{n}{i} u^i (1-u)^{n-i}\right].$$

Stwierdzenie (patrz Stwierdzenia 9 i 10) *Następujące oszacowania są optymalne.*
(i) Jeśli $n \geq 3$, to

$$\begin{aligned} 0 = \Psi_{1:n}(1) \leq & \quad \tfrac{\mathbb{V}ar(X_{2:n} - X_{1:n})}{\mathbb{V}ar\, X_1} & \leq \Psi_{1:n}(0) = n, \\ 0 = \Psi_{n-1:n}(0) \leq & \quad \tfrac{\mathbb{V}ar(X_{n:n} - X_{n-1:n})}{\mathbb{V}ar\, X_1} & \leq \Psi_{n-1:n}(1) = n. \end{aligned}$$

(ii) Jeśli $n \geq 4$ i $2 \leq i \leq n-2$, to pochodna $\Psi'_{i:n}$ ma albo jedno albo trzy miejsca zerowe, i wtedy

$$0 = \Psi_{i:n}(0) = \Psi_{i:n}(1) \leq \frac{Var(X_{i+1:n} - X_{i:n})}{\mathbb{V}ar\, X_1} \leq \Psi_{i:n}(u_0),$$

gdzie u_0 jest albo pojedynczym miejscem zerowym pochodnej albo pierwszym bądź trzecim miejscem zerowym $\Psi'_{i:n}$. Spośród nich wybierany jest ten argument, dla którego wartość $\Psi_{i:n}$ jest większa.

Przypadek $n = 2$ oraz $i = 1$, dla którego dolne oszacowanie jest dodatnie, nie jest omawiany w tym streszczeniu.

Wartości oczekiwane kombinacji liniowych rekordów

Niech $X_1, X_2, \ldots$ będą niezależnymi zmiennymi losowymi o tym samym rozkładzie ze skończoną średnią μ. Ponadto, niech $R_{1,k}, R_{2,k}, \ldots$ oznaczają odpowiednie wartości k-tych rekordów. Załóżmy, że $\mathbb{E}R_{n,k} < \infty$ dla pewnych ustalonych n i k. Poniżej opisujemy optymalne dolne i górne oszacowania wartości oczekiwanych kombinacji liniowych k-tych rekordów $\mathbb{E}\left[\sum_{i=1}^{n} c_i (R_{i,k} - \mu)\right]$, scentrowanych względem średniej populacji, i wyrażonych w jednostkach średniej różnicy Giniego $\Delta = \mathbb{E}|X_1 - X_2|$. Używamy następujących oznaczeń

$$\begin{aligned} \Xi_{n,k}(u) &= \frac{1}{2u}\left[(1-u)^{k-1}\sum_{i=0}^{n-1}\frac{\left[-k\ln(1-u)\right]^i}{i!} - 1\right], \\ \Xi_{\mathbf{c},k}(u) &= \sum_{i=1}^{n} c_i \Xi_{i,k}(u) = \frac{1}{2u}\left[(1-u)^{k-1}\sum_{i=0}^{n-1}\left(\sum_{j=i+1}^{n} c_j\right)\frac{[-k\ln(1-u)]^i}{i!} - \sum_{j=1}^{n} c_j\right], \end{aligned}$$

gdzie $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$.

Twierdzenie (patrz Twierdzenie 5) *Przy powyższych założeniach i oznaczeniach następujące oszacowania*

$$\inf_{0<u<1} \Xi_{\underline{c},k}(u) \leq \frac{\mathbb{E}\left[\sum_{i=1}^n c_i(R_{i,k} - \mu)\right]}{\Delta} \leq \sup_{0<u<1} \Xi_{\underline{c},k}(u)$$

są optymalne.

Warunki osiągalności są podobne do tych z problemów oszacowywania wartości oczekiwanych L-statystyk. Jedyna różnica polega na tym, że dokładne wartości w punktach 0 i 1 są zastąpione przez odpowiednie granice, a dyskretne rozkłady dwu-punktowe są zastąpione przez ich ciągłe przybliżenia. Zgodnie z powyższym, wszystkie oszacowania dla kombinacji rekordów są osiągane w granicy. Dla pojedynczych rekordów $R_{n,k}$, zachodzi $\Xi_{\mathbf{c},k} = \Xi_{n,k}$.

Stwierdzenie (patrz Stwierdzenie 12) *Dla różnych liczb naturalych $n \geq 2$ i $k \geq 1$, mamy następujące optymalne oszacowania.*
(i) Dla $n \geq 2$ i $k = 1$, zachodzi

$$\frac{1}{2} = \Xi_{n,1}(0+) \leq \frac{\mathbb{E}(R_{n,1} - \mu)}{\Delta} \leq \Xi_{n,1}(1-) = \infty.$$

(ii) Jeżeli $n = k = 2$, to

$$-\frac{1}{2} = \Xi_{2,2}(1-) \leq \frac{\mathbb{E}(R_{2,2} - \mu)}{\Delta} \leq \Xi_{2,2}(0+) = \frac{1}{2}.$$

(iii) Dla $n \geq 3$ i $k = 2$

$$-\frac{1}{2} = \lim_{u \nearrow 1^-} \Xi_{n,2}(1-) \leq \frac{\mathbb{E}(R_{n,2} - \mu)}{\Delta} \leq \Xi_{n,2}(u_1) > \frac{1}{2},$$

gdzie u_1 jest jedynym miejscem zerowym $\Xi'_{n,2}$ w przedziale $(0, 1)$.
(iv) Dla $n = 2$ oraz $k \geq 3$

$$-\frac{1}{2} > \Xi_{2,k}(u_1) \leq \frac{\mathbb{E}(R_{2,k} - \mu)}{\Delta} \leq \Xi_{2,k}(0+) = \frac{1}{2},$$

gdzie $u_1 \in (0, 1)$ jest jedynym miejscem zerowym $\Xi'_{2,k}$ in $(0, 1)$.
(v) Dla $k \geq 3$ i $n \geq 3$, mamy

$$-\frac{1}{2} > \Xi_{n,k}(u_2) \leq \frac{\mathbb{E}(R_{n,k} - \mu)}{\Delta} \leq \Xi_{n,k}(u_1) > \frac{1}{2},$$

gdzie $0 < u_1 < u_2 < 1$ są jedynymi rozwiązaniami równania $\Xi'_{n,k}(u) = 0$.

Podobnie wyznaczamy oszacowania wartości oczekiwanych różnic wartości k-tych rekordów $\mathbb{E}(R_{n,k} - R_{m,k})$, $1 \leq m < n$.

Oszacowania wariancji kombinacji liniowych rekordów

Niech $X_1, X_2, \ldots$ będą niezależnymi zmiennymi losowymi o tym samym rozkładzie i skończonym drugim momencie. Załóżmy też, że $\mathbb{E}R_{n,k}^2 < \infty$. Dla danych liczb naturalnych n i k, oraz dla ustalonego niezerowego wektora $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$, definiujemy funkcję

$$\begin{aligned}
\Phi_{\mathbf{c},k}(u,v) &= \frac{(1-v)^{k-1}}{u}\left\{\left[\sum_{j=1}^{n} c_j - (1-u)^k \sum_{i=0}^{n-1}\left(\sum_{j=i+1}^{n} c_j\right)\frac{[-k\ln(1-u)]^i}{i!}\right]\right.\\
&\times \sum_{i=0}^{n-1}\left(\sum_{j=i+1}^{n} c_j\right)\frac{[-k\ln(1-v)]^i}{i!}\\
&- \left.\sum_{1\le i<j\le n} c_i c_j \sum_{p=0}^{j-i-1}\sum_{q=0}^{p}\frac{(-1)^q[-k\ln(1-u)]^{i+q}[-k\ln(1-v)]^{p-q}}{(i-1)!q!(p-q)!(p+i)}\right\}
\end{aligned}$$

określoną na trójkącie $0 < u \le v < 1$, a także $\Psi_{\mathbf{c},k}(u) = \Phi_{\mathbf{c},k}(u,u)$, $0 < u < 1$.

Twierdzenie (patrz Twierdzenie 6) *Przy powyższych warunkach i oznaczeniach mamy*

$$\frac{\mathbb{V}ar(\sum_{i=1}^n c_i R_{i,k})}{\mathbb{V}ar\, X_1} \le \sup_{0<u\le v<1} \Phi_{\mathbf{c},k}(u,v).$$

Ponadto, jeśli

$$\sup_{0<u\le v<1} \Phi_{\mathbf{c},k}(u,v) = \sup_{0<u<1} \Psi_{\mathbf{c},k}(u),$$

to oszacowanie jest optymalne.

Dowodzimy także, że dolne oszacowania wariancji kombinacji liniowych k-tych rekordów są równe 0, za wyjątkiem przypadku $k = 1$ i $c_1 \neq 0$. Dla spacji k-tych rekordów $R_{m+1,k} - R_{m,k}$, funkcja $\Psi_{\mathbf{c},k}$ ma prostszą postać

$$\Psi_{m,k}(u) = \frac{[-k\ln(1-u)]^m(1-u)^{k-1}}{um!}\left[1 - \frac{(1-u)^k[-k\ln(1-u)]^m}{m!}\right].$$

Stwierdzenie (patrz Stwierdzenie 14) *Następujące oszacowania są optymalne.*
(i) Jeśli $k = 1$ i $m \ge 1$, to

$$\frac{\mathbb{V}ar(R_{m+1,1} - R_{m,1})}{\mathbb{V}ar\, X_1} \le \Psi_{m,1}(1-) = +\infty.$$

(ii) Jeżeli $m = 1$ *oraz* $k \geq 2$*, to*

$$\frac{\mathbb{V}ar(R_{2,k} - R_{1,k})}{\mathbb{V}ar\, X_1} \leq \Psi_{1,k}(0+) = k.$$

(iii) Jeśli albo $k = 2 \leq m$ *albo* $k \geq 3$ *oraz* $2 \leq m \leq \frac{2}{3}k$*, to*

$$\frac{\mathbb{V}ar(R_{m+1,k} - R_{m,k})}{\mathbb{V}ar\, X_1} \leq \Psi_{m,k}(u_0),$$

gdzie $0 < u_0 < 1$ *jest jedynym rozwiązaniem równania* $\Psi'_{m,k}(u) = 0$.
(iv) Jeżeli wreszcie $k \geq 3$ *oraz* $m > \frac{2}{3}k$*, to*

$$\frac{\mathbb{V}ar(R_{m+1,k} - R_{m,k})}{\mathbb{V}ar\, X_1} \leq \Psi_{m,k}(u_0),$$

gdzie $0 < u_0 < 1$ *argumentem, dla którego funkcja* $\Psi_{m,k}$ *osiąga swoje globalne maksimum na przedziale* $(0, 1)$.

W ostatnim przypadku, formalnie możemy jedynie udowodnić, że $\Psi_{m,k}$ ma co najwyżej trzy lokalne maksima.